ALGEBRAIC K-THEORY OF MAPPING CLASS GROUPS

ETHAN BERKOVE, DANIEL JUAN-PINEDA, AND QIN LU

Abstract. We show that the Fibered Isomorphism Conjecture of T. Farrell and L. Jones holds for various mapping class groups. In many cases, we explicitly calculate the lower algebraic \( K \)-groups, showing that they do not always vanish.

1. Introduction

Let \( \Gamma \) be a torsion-free discrete group. It is a well-known conjecture that the Whitehead group \( Wh(\Gamma) \) of \( \Gamma \) must vanish. A major tool in the pursuit of this conjecture has been the Fibered Isomorphism Conjecture (FIC) of T. Farrell and L. Jones. The FIC asserts that the algebraic \( K \)-theory groups of \( \mathbb{Z} \Gamma \) may be computed from the corresponding algebraic \( K \)-theory groups of the virtually cyclic subgroups of \( \Gamma \) (see [12] or the Appendix for a precise formulation). When FIC holds for a torsion-free group, its Whitehead group vanishes.

FIC has been verified in many instances: for discrete cocompact subgroups of virtually connected Lie groups by Farrell and Jones [12]; for pure braid groups by Aravinda, Farrell, and Roushon [1]; for braid groups by Farrell and Roushon [11]; for finitely generated Fuchsian groups by Berkove, Juan-Pineda, and Pearson [2]; and for Bianchi groups by Berkove, Juan-Pineda, Farrell, and Pearson [3]. In this paper we investigate when FIC holds for various mapping class groups.

The pure mapping class group \( \Gamma^i_{pr} \) is the group of path components of orientation preserving self-diffeomorphisms of an orientable surface of genus \( g \), with \( i \) punctures and \( r \) boundary components. We require that these diffeomorphisms pointwise fix the punctures and are the identity on the boundary components. The full mapping class group is defined similarly, but includes diffeomorphisms which permute the punctures.

The techniques in this paper build on results in [1] and [11] concerning strongly poly-free groups. Although the mapping class groups are not themselves strongly poly-free, they admit descriptions where strongly poly-free groups figure prominently. We prove in this paper that FIC holds for all pure mapping class groups of genus 0, 1 and 2 and for all full mapping class groups of genus 0 and 2.

**Theorem 5.1.** Let \( \Gamma \) be a torsion-free subgroup of any mapping class group, pure or full, for which FIC holds. Then \( \tilde{K}_i(\mathbb{Z}\Gamma) = 0 \) for all \( i \leq 1 \).

When a mapping class group is torsion-free, Theorem 5.1 implies that its Whitehead group is trivial. However, many mapping class groups contain considerable torsion. In these cases, when FIC holds for a group, often an explicate calculation is possible. For example, we have a non-vanishing result for algebraic \( K \)-groups. We note that

\[
\tilde{K}_i(\mathbb{Z}\Gamma) =
\begin{cases}
Wh(\Gamma), & \text{if } i = 1; \\
\tilde{K}_0(\mathbb{Z}\Gamma), & \text{if } i = 0; \\
K_i(\mathbb{Z}\Gamma), & \text{if } i < 0.
\end{cases}
\]

**Theorem 5.8.** Let \( \Gamma \) be a pure mapping class group of genus \( g = 1 \). Then for all \( i \leq 1 \), \( \tilde{K}_i(\mathbb{Z}\Gamma) = 0 \) with two exceptions: when \( \Gamma = \Gamma_1^0 \) or \( \Gamma_1^1 \), there is one non-vanishing \( K \)-group, namely \( K_{-1}(\mathbb{Z}\Gamma) = \mathbb{Z} \).

The techniques we develop in the paper also allow us to study fundamental groups of configuration spaces.

This paper is organized as follows. In section 2, we give the basic techniques and results that allow us to apply FIC to mapping class groups. In section 3 we prove FIC for various pure mapping class groups. We develop these techniques further in section 4 to study the case of full mapping class groups. We then use the implications of FIC to perform explicit \( K \)-theory calculations in section 5. In section 6 we mention how our techniques apply to other groups, particularly the fundamental groups of configuration spaces. We mention in the appendix the setup for FIC and some immediate consequences of its validity.

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2. AN EXTENSION OF THE FIBERED ISOMORPHISM CONJECTURE

Before we state our extension of the theorem, we introduce some background material. We start with two classes of groups that figure prominently in our analysis.

**Definition 2.1.** A group is called *virtually cyclic* if it contains a cyclic group of finite index.

In particular, a finite group is virtually cyclic. All infinite virtually cyclic groups contain an infinite cyclic group of finite index. Our arguments also involve a particular type of torsion-free group whose definition appears in a paper by Aravinda, Farrell and Roushon.

**Definition 2.2.** [1] A discrete group $\Gamma$ is called *strongly poly-free* if there exists a finite filtration by subgroups $1 = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_n = \Gamma$ such that the following conditions are satisfied:

1. $\Gamma_i$ is normal in $\Gamma$ for each $i$.
2. $\Gamma_{i+1}/\Gamma_i$ is a finitely generated free group for all $i$.
3. For each $\gamma \in \Gamma$ there is a compact surface $F$ and a diffeomorphism $f : F \to F$ such that the induced homomorphism $f_\#$ on $\pi_1(F)$ is equal to $c_\gamma$ in $Out(\pi_1(F))$, where $c_\gamma$ is the action of $\gamma$ on $\Gamma_{i+1}/\Gamma_i$ by conjugation and $\pi_1(F)$ is identified with $\Gamma_{i+1}/\Gamma_i$ via a suitable isomorphism.

The third condition says that the algebraic action of $\gamma$ on $\Gamma_{i+1}/\Gamma_i$ can be geometrically realized.

**Remark 2.3.** It is implicit in the Appendix of [11] that the finite product of strongly poly-free groups is also strongly poly-free.

Our results build on a few known theorems. The first two are theorems of Farrell and Jones.

**Theorem 2.4.** [12, Proposition 2.2] Let $p : \Gamma \to G$ be an epimorphism of groups such that FIC is true for $G$ and $p^{-1}(S)$ where $S$ ranges over all virtually cyclic subgroups of $G$. Then FIC is also true for $\Gamma$.

**Theorem 2.5.** [12, A.8] If FIC holds for a group $\Gamma$, then FIC also holds for all subgroups of $\Gamma$.

We will also use Farrell and Roushon’s Main Theorem, which applies to extensions by strongly poly-free groups.

**Theorem 2.6.** [11] Let $\Gamma$ be an extension of a finite group by a strongly poly-free group (the finite group is the quotient group). Then $\Gamma$ satisfies FIC.
We combine these two results in a theorem that allows us to apply FIC to the punctured mapping class groups.

**Theorem 2.7.** Say that $\Gamma$ fits into an extension $1 \to K \to \Gamma \to G \to 1$, where FIC holds for $G$ and $K$ is a finitely generated free group. Furthermore, assume for all $t \in G$ of infinite order, that the action of the lift $\hat{t}$ on $K$ can be geometrically realized. Then $\Gamma$ satisfies FIC.

**Proof.** Let $p : \Gamma \to G$ be the epimorphism in the short exact sequence. Given any virtually cyclic subgroup $S \subseteq G$, we will show that FIC holds for $\hat{S} = p^{-1}(S)$.

For all virtually cyclic subgroups $S$ there is an extension

$$1 \to K \to \hat{S} \xrightarrow{p} S \to 1.$$ 

We consider two cases: when $S$ is a finite group and when it is not. In the case where $|S| < \infty$, Theorem 2.6 directly implies that FIC holds for $\hat{S}$. Next, assume that $S$ is an arbitrary infinite virtually cyclic group, and let $T$ be a cyclic subgroup of finite index in $S$ with generator $t$. Note that we may choose $\hat{T}$ so it is normal in $\hat{S}$. Pick a $\hat{t}$ such that $p(\hat{t}) = t$; the subgroup $\langle K, \hat{t} \rangle \subseteq \hat{S}$ is normal in $\hat{S}$ as $T$ is normal in $S$.

We claim that $\langle K, \hat{t} \rangle$ is a strongly poly-free group. The extension

$$1 \to K \to \langle K, \hat{t} \rangle \to \mathbb{Z} \to 1$$

identifies $\langle K, \hat{t} \rangle$ as the semidirect product $K \rtimes \langle \hat{t} \rangle$, which is filtered by $1 = \Gamma_0 \subseteq K \subseteq K \rtimes \langle \hat{t} \rangle = \Gamma_2$. Thus, the first two conditions for a strongly poly-free group are satisfied. The third condition is satisfied by our assumption of the action of $\hat{t}$ on $K$.

We again write $\hat{S}$ as a finite extension, this time with strongly poly-free kernel $K \rtimes \langle \hat{t} \rangle$:

$$1 \to K \rtimes \langle \hat{t} \rangle \to \hat{S} \to F \to 1.$$ 

Now, Theorem 2.6 implies that FIC holds for $\hat{S}$. As $S$ is arbitrary, Theorem 2.4 implies the result. \hfill $\square$

**Remark 2.8.** The geometric realization condition in Definition 2.2 is necessary. It is not difficult to find algebraic actions which have no geometric realization (see, for example, Stallings [22]). At this point it is not known if FIC holds in general for cases where geometric realization is not possible.

We mention one other way that FIC can be extended to new groups.

**Lemma 2.9.** Given a short exact sequence

$$1 \to F \to \Gamma \xrightarrow{p} Q \to 1$$

...
with $F$ finite, if FIC holds for $Q$ then it also holds for $\Gamma$.

Proof. Let $S$ be any virtually cyclic subgroup of $Q$, and let $T \subseteq S$ be a cyclic group of finite index, possibly all of $S$. There is a short exact sequence

$$1 \to F \to p^{-1}(T) \to T \to 1.$$ 

By definition, $p^{-1}(T)$ is a virtually cyclic group, so it contains a cyclic subgroup $\hat{T}$ of finite index. Furthermore, $p^{-1}(T)$ is of finite index in $p^{-1}(S)$, which implies that $p^{-1}(S)$ is virtually cyclic as it also contains $\hat{T}$. Therefore, FIC holds for $p^{-1}(S)$. Theorem 2.4 completes the proof as $S$ is arbitrary. \qed

3. Pure Mapping Class Groups

Denote by $S_{g,r}^i$ an orientable surface of genus $g$ with $i$ punctures and $r$ boundary components. We define $Diff^+(S_{g,r}^i)$ to be the group of orientation-preserving diffeomorphisms of $S_{g,r}^i$ which pointwise fix the punctures and the boundary. Furthermore, let $Iso(S_{g,r}^i)$ be the subgroup of $Diff^+(S_{g,r}^i)$ consisting of all diffeomorphisms that are isotopic to the identity map. We define the pure mapping class group, $\Gamma_{g,r}^i$, as the quotient $Diff^+(S_{g,r}^i)/Iso(S_{g,r}^i)$.

In a more general construction, one can define $Diff^+(S_{g,r}^i)$ as the set of diffeomorphisms which fix the set of punctures, possibly permuting them. The group constructed in this manner is called the full mapping class group, which we denote by $\hat{\Gamma}_{g,r}^i$. The full mapping class group is a finite extension of the pure mapping class group

$$(3.1) \quad 1 \to \Gamma_{g,r}^i \to \hat{\Gamma}_{g,r}^i \to \Sigma_i \to 1,$$

where $\Sigma_i$ is the symmetric group on $i$ letters. When $i > 0$ and $r = 0$, we refer to the mapping class group as $\Gamma_{g,r}^i$ and when both $i$ and $r$ equal 0, we write $\Gamma_g$ to denote the unpunctured mapping class group.

There are two short exact sequences [14] which tie together different members of the mapping class family. These sequences allow us to extend FIC from group to group. They are

$$(3.2) \quad 1 \to \pi_1(S_{g,r}^i) \to \Gamma_{g,r}^{i+1} \to \Gamma_{g,r}^i \to 1 \quad \text{if } 2g + r + i > 2,$$

$$(3.3) \quad 1 \to \mathbb{Z} \to \Gamma_{g,r+1}^{i-1} \to \Gamma_{g,r}^i \to 1 \quad \text{if } 2g + 2r + i > 2.$$ 

Note that these sequences relate mapping class groups of the same genus.

**Theorem 3.1.** Assume $2g + r + i > 2$ and $r + i > 0$. If FIC holds for $\Gamma_{g,r}^i$, then it also holds for (1) $\Gamma_{g,r}^{i+1}$; and for (2) $\Gamma_{g,r+1}^{i-1}$ when $i > 0$. 

Proof. Case (1): Since $2g + r + i > 2$, we can use Sequence 3.2. As $r + i > 0$, $\pi_1(S^i_{g,r})$ is a finitely generated free group, so Theorem 2.7 implies the result once we show the appropriate geometric realizability condition. This follows as elements of mapping class groups are intrinsically geometric. Specifically, take $g \in \Gamma^i_{g,r}$ any element of infinite order, and let $\hat{g}$ be any lift. As $\hat{g} \in \Gamma^{i+1}_{g,r}$, it can be represented by a self-diffeomorphism of $S^i_{g,r}$. By ignoring the appropriate puncture, it is also a self-diffeomorphism of $S^i_{g,r}$ inducing the appropriate action.

Case (2): Use Sequence 3.3. As in Case (1), we look at the action of $\hat{g}$ on the strongly poly-free group $\mathbb{Z}$. There are only two possibilities, the trivial and the involution actions, both of which are geometrically realizable on a cylinder.

\begin{theorem}
FIC holds for all pure mapping class groups $\Gamma^i_{0,r}$.
\end{theorem}

Proof. It suffices to show that FIC holds for all mapping class groups $\Gamma^i_{0,r}$, with $r + i \leq 3$. Then we can use Theorem 3.1 and induction on $i$, then $r$, to show that FIC holds for the rest of the cases.

It is well-known that $\Gamma^i_0 \cong 1$ for $0 \leq i \leq 3$, so FIC holds trivially for these groups. Also, $\Gamma^0_{0,r} \cong P_{r-1} \times \mathbb{Z}^{r-1}$, where $P_r$ is the pure braid group on $r$ strings [5, 14]. Aravinda, Farrell and Roushon prove in [1] that FIC holds for $P_r \times \mathbb{Z}^k$ for any $k$.

Applying Case (2) of Theorem 3.1 to $\Gamma^3_0$ implies that FIC holds for $\Gamma^2_{0,1}$. The same argument implies that FIC holds for $\Gamma^1_{0,2}$ as $2g + 2r + i = 4$. For the case $\Gamma^1_{0,1}$, we start with the observation that $\Gamma^0_{0,2} \cong \mathbb{Z}$. This follows as $\Gamma^0_{0,2} \cong P_1 \times \mathbb{Z} \cong \mathbb{Z}$. Thus, Sequence 3.3 becomes

$$1 \to \mathbb{Z} \to \mathbb{Z} \to \Gamma^1_{0,1} \to 1,$$

which implies that FIC holds for $\Gamma^1_{0,1}$ as it is finite. (In fact, as $r > 0$, we shall see that this group is torsion-free, hence trivial.)

\begin{theorem}
FIC holds for all pure mapping class groups $\Gamma^i_{1,r}$.
\end{theorem}

Proof. We only need consider the cases $r + i \leq 1$ in order to apply Theorem 3.1. First, $\Gamma^0_{1} \cong \Gamma^1_{1} \cong SL_2(\mathbb{Z}) \cong \mathbb{Z}/4 \ast \mathbb{Z}/2 \mathbb{Z}/6$. As $SL_2(\mathbb{Z})$ acts on $\mathbb{H}^2$, hyperbolic 2-space, with finite area quotient, results in [3] imply that FIC holds for this group.

The final case, $\Gamma^0_{1,1}$, now follows from Case (2) of Theorem 3.1 with $i = 1, r = 0$.

We have to prove that FIC holds for mapping class groups one genus at a time since the value of the genus stays fixed within Sequences 3.2 and 3.3. In general, one wants to show for a fixed value of $g$ that FIC
holds for a mapping class group whose values of \( i \) and \( r \) are as small as possible.

**Remark 3.4.** In [21], Roushon defines a **strongly poly-surface** group. This is similar to a strongly poly-free group, except that \( \Gamma_{i+1}/\Gamma_i \) is isomorphic to the fundamental group of a surface. (There is also a technical condition that is always satisfied when the surface is closed.)

In the proof of Roushon’s Main Theorem, he implies that FIC holds for any extension

\[ 1 \to K \to \Gamma \to G \to 1, \]

where \( G \) is finite and \( K \) is a strongly poly-surface group. Using this result we get a version of Theorem 2.7 with “finitely generated free group” replaced by “fundamental group of a surface.” The proof is identical.

**Theorem 3.5.** Assume \( g \geq 2 \). If FIC holds for a mapping class group \( \Gamma_g \), then FIC holds for \( \Gamma_{g,r}^i \) for all \( r \) and \( i \).

**Proof.** Using the unpunctured mapping class group, Sequence 3.2 becomes

\[ 1 \to \pi_1(S_g) \to \Gamma_g^1 \to \Gamma_g \to 1 \]

where \( S_g \) is a compact surface of genus \( g \). As \( \pi_1(S_g) \) is the fundamental group of a surface, the version of Theorem 2.7 mentioned in Remark 3.4 implies that FIC holds for \( \Gamma_g^1 \). Then inductively apply Theorem 3.1. \( \square \)

### 4. Full Mapping Class Groups

We would like to show that FIC holds for the hyperelliptic mapping class groups and the unpunctured mapping class group \( \Gamma_2 \). One route to this result is to work with the full mapping class groups of genus 0, which are closely related to these objects. Sequence 3.1 shows that the full mapping class groups contain the pure mapping class groups as subgroups of finite index. Farrell and Roushon develop techniques in [11] that are appropriate to such cases, and we adapt them to mapping class groups.

**Definition 4.1.** Given groups \( K \) and \( Q \), with \( Q \) finite, the **wreath product** of \( K \) and \( Q \) is the group \( K \wr Q = K^{|Q|} \rtimes Q \) where \( K^{|Q|} \) is the product of \(|Q|\) copies of \( K \) indexed by elements of \( Q \), and \( Q \) acts on \( K^{|Q|} \) via the regular action of \( Q \) on \( Q \).

**Theorem 4.2.** [9] Take a sequence \( 1 \to K \to G \to Q \to 1 \) with \( Q \) a **finite group**. Then there is an injective homomorphism \( G \to K \wr Q \).
Recall that when FIC holds for a group it also holds for any subgroup (this is Theorem 2.5). In light of Theorem 4.2 and Sequence 3.1, to show that FIC holds for $\tilde{\Gamma}_0^i$ it suffices to show that FIC holds for $\Gamma_0^i \lhd \Sigma_i$. We will prove a stronger statement.

**Theorem 4.3.** Let $Q$ be a finite group. Then FIC holds for $\Gamma_0^i \lhd Q$ for all $i$.

**Proof.** We proceed by induction. We note that the theorem holds trivially for $i = 0, 1, 2, 3$, as $\Gamma_0^i \cong 1$ for these cases.

Let $Q$ be any finite group. Since $\pi_1(S_0^i)$ is a normal subgroup of $\Gamma_0^{i+1}$, $\pi_1(S_0^i)^{|Q|}$ is a normal subgroup of $\Gamma_0^{i+1} \lhd Q$. Therefore, there is another short exact sequence that comes from Sequence 3.2

$$1 \to \pi_1(S_0^i)^{|Q|} \to \Gamma_0^{i+1} \lhd Q \xrightarrow{p} \Gamma_0^i \lhd Q \to 1.$$ 

By the induction assumption, FIC holds for $\Gamma_0^i \lhd Q$. Now let $S$ be any virtually cyclic subgroup of $\Gamma_0^i \lhd Q$. If we can show that FIC holds for all groups $p^{-1}(S)$, then we have that FIC holds for $\Gamma_0^{i+1} \lhd Q$.

There are two cases to consider, when $|S| < \infty$ and when $S$ is infinite virtually cyclic. In the finite case, there is a short exact sequence

$$1 \to \pi_1(S_0^i)^{|Q|} \to p^{-1}(S) \to S \to 1.$$ 

The group $\pi_1(S_0^i)^{|Q|}$ is strongly poly-free by Remark 2.3, hence FIC holds for $p^{-1}(S)$ by Theorem 2.6.

The second case is when $S$ is an infinite group. Let $t$ be a generator of a normal infinite cyclic subgroup of finite index in $S$. The element $t$ lifts to an element $\hat{t} \in \Gamma_0^{i+1} \lhd Q$. By taking a larger power of $t$ if necessary, we can assume that $\hat{t} \in (\Gamma_0^{i+1})^{|Q|}$, that is, it acts component-by-component. As in the proof of Theorem 2.7, there is a new short exact sequence

$$1 \to \pi_1(S_0^i)^{|Q|} \rtimes < \hat{t} > \to p^{-1}(S) \to Q' \to 1$$ 

where $Q'$ is a finite group. Notice that $\pi_1(S_0^i)^{|Q|} \rtimes < \hat{t} >$ is a subgroup of $\prod_{|Q|} \pi_1(S_0^i) \rtimes < \hat{t} >$, where the semidirect product action is considered by coordinate. (This is Fact 2.4 from [11].) Therefore, $p^{-1}(S)$ is a subgroup of a group $\Gamma'$ which fits into an extension

$$1 \to \prod_{|Q|} \pi_1(S_0^i) \rtimes < \hat{t} > \to \Gamma' \to Q' \to 1.$$ 

Each term in the product, $\pi_1(S_0^i) \rtimes < \hat{t} >$, is a strongly poly-free group by Theorem 2.7, so by Remark 2.3, $\prod_{|Q|} \pi_1(S_0^i) \rtimes < \hat{t} >$ is also a strongly poly-free group. At this point, Theorem 2.6 implies that FIC holds for $\Gamma'$; the subgroup inheritance property in Theorem 2.5 then
implies that FIC also holds for $p^{-1}(S)$. This completes the induction step and the proof.

**Corollary 4.4.** Let $G$ be a group that fits into an extension $1 \rightarrow \Gamma_0^i \rightarrow G \rightarrow Q \rightarrow 1$ with $|Q| < \infty$. Then FIC holds for $G$. In particular, FIC holds for the full mapping class group $\hat{\Gamma}_0^i$.

**Proof.** The first claim follows from Theorem 4.2 and Theorem 4.3 as $G \subseteq \Gamma_0^i \not\subseteq Q$. The second claim follows from Sequence 3.1.

Given an unpunctured surface of genus $g > 0$, there is special diffeomorphism, the hyperelliptic involution, which acts as a reflection across all of the holes. The normalizer of this involution in $\Gamma_g$ is the hyperelliptic mapping class group $\Delta_g$. There is a relationship between the hyperelliptic mapping class groups and the full punctured mapping class groups of genus 0 given by the sequence

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \Delta_g \rightarrow \hat{\Gamma}_0^{2g+2} \rightarrow 1.$$ 

It is well-known that $\Delta_2 = \Gamma_2$ [5]. In general, however, $\Delta_g$ is neither normal nor of finite index in $\Gamma_g$. For more information on the hyperelliptic mapping class groups, the reader should consult [6] or [13].

**Corollary 4.5.** FIC holds for the hyperelliptic mapping class groups.

**Proof.** Apply Theorem 2.9 to the short exact sequence

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \Delta_g \rightarrow \hat{\Gamma}_0^{2g+2} \rightarrow 1.$$ 

**Corollary 4.6.** FIC holds for all pure mapping class groups of genus 2.

**Proof.** FIC holds for $\Gamma_2 = \Delta_2$. Thus, by Theorem 3.5, FIC holds for $\Gamma_2^{i_r}$.

As in the case of the pure mapping class groups, one must proceed genus-by-genus to show that FIC holds for the full mapping class groups. Theorem 4.3 is valid for any genus, as long as Sequence 3.2 holds and an ad hoc calculation for the base case of the induction is performed.

**Example 4.7.** We will show that FIC holds for all full mapping class groups of genus 2. In order to use Theorem 4.3, we must first show that FIC holds for $\Gamma_2 \not\subseteq Q$, where $Q$ is any finite group.

We start with the short exact sequence

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \Gamma_2 \rightarrow \hat{\Gamma}_0^6 \rightarrow 1.$$
As in the proof of Theorem 4.3 we can build another short exact sequence

$$1 \to (\mathbb{Z}/2)^{|Q|} \to \Gamma_2 \wr Q \to \hat{\Gamma}_0 \wr Q \to 1$$

with finite kernel. Hence, by Lemma 2.9, if FIC holds for $\hat{\Gamma}_0 \wr Q$ it will also hold for $\Gamma_2 \wr Q$. We direct our attention to $\hat{\Gamma}_0 \wr Q$.

We have another short exact sequence based on Sequence 3.1:

$$1 \to (\Gamma_0^6)^{|Q|} \to \hat{\Gamma}_0 \wr Q \to \Sigma_6 \wr Q \to 1.$$  

Theorem 4.2 implies that $\hat{\Gamma}_0^6 \wr Q \subseteq (\Gamma_0^6)^{|Q|} \wr (\Sigma_6 \wr Q)$. Furthermore, $(\Gamma_0^6)^{|Q|} \wr (\Sigma_6 \wr Q) \subseteq \Gamma_0^6 \wr (\Sigma_6 \wr Q \times Q)$ by Fact 2.4 of [11]. Since the group $(\Sigma_6 \wr Q \times Q)$ is finite, FIC holds for $\Gamma_0^6 \wr (\Sigma_6 \wr Q \times Q)$ by Theorem 4.3.

Therefore, FIC holds for $\hat{\Gamma}_0^6 \wr Q$ by subgroup inheritance, which in turn implies that FIC holds for $\Gamma_2 \wr Q$. We have now completed a base case for an inductive argument. However, we cannot apply Theorem 4.3, because in Sequence 3.2

$$1 \to \pi_1(S_2) \to \Gamma_2^1 \to \Gamma_2 \to 1,$$

$\pi_1(S_2)$ is not a free group. However, $\pi_1(S_2)$ is a strongly poly-surface group, and the proof of Theorem 4.3 is valid for this case by an identical argument. This shows FIC is true for $\Gamma_2^1 \wr Q$, at which point Theorem 4.3 applies directly.

5. Calculations

In this section we explore consequences of the validity FIC as they apply to the calculation of lower algebraic $K$-groups. Our first theorem is a vanishing result.

**Theorem 5.1.** Let $\Gamma$ be a torsion-free subgroup of any mapping class group, pure or full, for which FIC holds. Then $\tilde{K}_i(\mathbb{Z}\Gamma) = 0$ for all $i \leq 1$.

**Proof.** This follows from Remark 7.1 in the Appendix. □

**Remark 5.2.** From results in the previous sections, groups that satisfy the hypotheses of the above theorem include

1. $\Gamma_{0,1}^g$ for $g = 0,1$ and $r > 0$ as these groups are torsion-free,
2. $\Gamma_{0,r}^1$ as these are products of pure braid groups and torsion-free abelian groups,
3. any torsion-free subgroup of any pure mapping class group of genus $\leq 2$, and any torsion-free subgroup of any full mapping class group of genus 0 or 2.
(4) $\Gamma_0$. Recall that $\Gamma_0^0 \cong \Gamma_1^0 \cong \Gamma_2^0 \cong \Gamma_3^0 \cong 1$. Therefore, $\Gamma_0^i$ is torsion-free for $i > 3$ as both the kernel and quotient of Sequence 3.2 are torsion-free.

When we want to calculate the lower algebraic K-theory of a mapping class group which contains torsion there is more work to do. A calculation of the lower algebraic K-theory of a given mapping class group consists of three steps. First, we show that FIC holds for the group. Second, we classify the group’s virtually cyclic subgroups. Finally, we perform the K-theory calculation, using a classifying space for a suitable family of subgroups of the mapping class group. We note, however, that the unpunctured mapping class groups may possibly contain subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}/p \times \mathbb{Z}/p$. The algebraic K-theory of these infinite virtually cyclic groups is infinitely generated, so the lower algebraic K-theory of the unpunctured mapping class group will be a difficult object to calculate in these cases! When $i \geq 1$, $\mathbb{Z}/p \times \mathbb{Z}/p$ is not a subgroup of the mapping class group by results in [15], so a complete lower algebraic K-theory calculation seems more reasonable.

Using structure theorems for the punctured pure mapping class groups, it is possible to determine which virtually cyclic groups can appear as subgroups. There is a helpful description of infinite virtually cyclic groups due to Maskit in [18].

**Theorem 5.3.** An infinite virtually cyclic group $G$ fits into one of the two following short exact sequences, with $F$ a finite group.

$$1 \to F \to G \to \mathbb{Z} \to 1$$

or

$$1 \to F \to G \to D_\infty \to 1.$$  

In the former case, $G$ is the semidirect product $F \rtimes \mathbb{Z}$. In the latter case, $G$ is the amalgamated product $G_1 \ast_F G_2$, where $F$ is an index two subgroup in both $G_1$ and $G_2$.

This characterization yields an effective way to classify infinite virtually cyclic subgroups when $i > 0$. One first classifies a group’s finite subgroups. Any virtual cyclic subgroup which contains a finite group will sit inside the finite group’s normalizer. Therefore, normalizers will be our next object of study. It is proven in [15] that $\Gamma_g^1$ has $p$-periodic cohomology, which implies that all finite groups contained in $\Gamma_g^1$ are cyclic. Since the kernel of the Sequence 3.2 is torsion-free, the same result applies for $\Gamma_g^i$.

**Corollary 5.4.** Let $G$ be a virtually cyclic subgroup of a punctured pure mapping class group. The following is a list of possible $G$: 
(1) $G$ is a finite cyclic group.
(2) $G$ is the direct product $\mathbb{Z}/p \times \mathbb{Z}$ with $p$ prime.
(3) $G$ is the semidirect product $\mathbb{Z}/n \rtimes \mathbb{Z}$. The action of the infinite
cyclic generator on all prime order cyclic subgroups of $\mathbb{Z}/n$ is
trivial.
(4) $G$ is the amalgamated product $\mathbb{Z}/2n \ast_{\mathbb{Z}/n} \mathbb{Z}/2n \cong \mathbb{Z}/n \times D_\infty$, with $n$ odd.

Proof. For finite groups, the result in [15] applies. Next assume that
$G$ has infinite order. Take a finite cyclic subgroup $\mathbb{Z}/n$ of a mapping
class group $\Gamma_g$ and consider its normalizer. It is an important fact
that when $n$ is prime $N_{\Gamma_g}(\mathbb{Z}/n) = C_{\Gamma_g}(\mathbb{Z}/n)$, i.e., the normalizer and
centralizer agree. Therefore, any extension of $\mathbb{Z}$ by a prime order cyclic
group must be a direct product, proving (2). On the other hand, an
extension of $\mathbb{Z}$ by a composite order cyclic group will split, yielding
$\mathbb{Z}/n \rtimes \mathbb{Z}$. Extensions of $\mathbb{Z}$ by prime cyclics appear as subgroups of
this semidirect product, implying (3). To prove (4), take $G$ of the
form $G_1 \ast_{\mathbb{Z}/n} G_2$. The only finite subgroup of $\Gamma_g$ which contains $\mathbb{Z}/n$
as an index two subgroup is $\mathbb{Z}/2n$. Also, $n$ must be odd; otherwise,
$\mathbb{Z}/n \times D_\infty$ contains a copy of $\mathbb{Z}/2 \times \mathbb{Z}/2$. We note that the groups $\mathbb{Z}$
and $D_\infty$ appear as degenerate examples of cases (3) and (4). □

We saw that when $g = 0$ the lower algebraic $K$-theory for pure
mapping class groups vanishes. The pure mapping class groups with
$g = 1$ provide more interesting and less trivial examples. We will use
two results to aid in our calculations: Nielsen’s Realization Theorem,
which states that an element of $\Gamma_g$ of finite order can be realized as
a diffeomorphism of $S_g$ of the same order; and the Riemann Hurwitz
equation. The Riemann Hurwitz equation uses the notion of a singular
point, which is defined as follows: Given any element of order $n$ in a
mapping class group $\Gamma_g$, represent it by a diffeomorphism $f$ of $S_g$
of order $n$ which fixes the $i$ punctures. There is a projection $\pi$ sending
$S_g$ to $S_h = S_g/\langle f \rangle$ which is a $n$-sheet branched covering. Let $\hat{P}$
be a point of $S_g$ that is fixed by some power of $f$. Specifically, there
is a largest value $k$ such that $f^{n/k}$ fixes $\hat{P}$. We say that the point
$P = \pi(\hat{P})$ in the quotient space $S_h$ is a singular point of order $k$ with
respect to the $\mathbb{Z}/n$ action. Such a singular point has $n/k$ preimages,
$f(\hat{P}), f^2(\hat{P}), \ldots, f^{n/k}(\hat{P}) = \hat{P}$, in $S_g$ which correspond to a single orbit
under $\langle f \rangle$. Note that a fixed point is the singular point of order $n$.

Assume that after projection, the branched covering $S_g \to S_h$ has
$q$ singular points $\{P_1, \ldots, P_q\}$, and that the order of $P_i$ is $k_i$. The
Riemann Hurwitz equation connects the genus of the surface, the genus
of its quotient space, the order of \( f \), and the singular point information together:

\[
2g - 2 = n \left( (2h - 2) + \sum_{i=1}^{g} \left( 1 - 1/k_i \right) \right)
\]

The orders of the torsion elements contained in the mapping class groups of genus 1 are known, and are summarized below.

**Lemma 5.5.** [16] If \( \Gamma_i^1 \) has \( p \)-torsion, then \( p = 2, 3 \).

1. \( \Gamma^1_1 \) has 2, 3 torsion.
2. \( \Gamma^1_2 \) has 2, 3 torsion.
3. \( \Gamma^1_3 \) has 2, 3 torsion.
4. \( \Gamma^1_4 \) has 2 torsion.
5. \( \Gamma^1_i \) has no \( p \)-torsion for \( i \geq 5 \).

We use this torsion information to classify the infinite virtually cyclic subgroups.

**Theorem 5.6.** For punctured mapping class groups of genus 1,

1. \( \Gamma^0 \cong \Gamma^1_1 \cong \text{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4 \ast_{\mathbb{Z}/2} \mathbb{Z}/6 \) and the possible virtually cyclic subgroups in \( \Gamma^1_1 \) are \( \{1, \mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/3, \mathbb{Z}/6, \mathbb{Z}, \mathbb{Z}/2 \times \mathbb{Z}, \mathbb{Z}/4 \times \mathbb{Z}, \mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z} \} \).
2. The possible virtually cyclic subgroups in \( \Gamma^1_2 \) are \( \{1, \mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/3, \mathbb{Z}/2 \times \mathbb{Z}, \mathbb{Z}/4 \times \mathbb{Z}, \mathbb{Z}/2 \times \mathbb{Z} \} \).
3. The possible virtually cyclic subgroups in \( \Gamma^1_3 \) are \( \{1, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}, \mathbb{Z}/2 \times \mathbb{Z}, \mathbb{Z}/4 \times \mathbb{Z} \} \).
4. The possible virtually cyclic subgroups in \( \Gamma^1_4 \) are \( \{1, \mathbb{Z}/2, \mathbb{Z}, \mathbb{Z}/2 \times \mathbb{Z} \} \).
5. The possible virtually cyclic subgroups in \( \Gamma^1_i \) for \( i \geq 5 \) are \( \{1, \mathbb{Z} \} \).

**Proof.** We will work with the mapping class group \( \Gamma^1_1 \) only, as the other cases are similar and easier. By Lemma 5.5, we know that \( \Gamma^1_1 \) has only 2 and 3 torsion, but we still need to determine the exponent of that torsion. Recall that \( \mathbb{Z}/p \times \mathbb{Z}/p \) does not appear as a subgroup of any punctured mapping class group.

**The 2-torsion exponent:** We claim that \( \mathbb{Z}/8 \) is not a subgroup of \( \Gamma^1_1 \). By way of contradiction, assume that there is a copy of \( \mathbb{Z}/8 \) in \( \Gamma^1_1 \). By Nielsen’s Realization Theorem, we can lift \( \mathbb{Z}/8 \) in \( \text{Diff}^+ (S^1_1) \). View \( \mathbb{Z}/8 \) as acting on \( S^1_1 \) with quotient space \( S_n \). This action has at least one fixed point, namely, the puncture in \( S^1_1 \). Therefore, the Riemann Hurwitz equation

\[
2g - 2 = p^3(2h - 2) + p^3(1 - 1/p)a_1 + p^3(1 - 1/p^2)a_2 + p^3(1 - 1/p^3)a_3
\]
must have a non-negative solution \((a_1, a_2, a_3)\), with \(a_3 \neq 0\). The value of \(a_3\) will give the number of fixed points of the \(\mathbb{Z}/8\) action. Substitute \(g = 1\) and \(p = 2\) into the Riemann Hurwitz equation to get

\[
0 = 8(2h - 2) + 4a_1 + 6a_2 + 7a_3.
\]

If \(h \geq 1\), \((a_1, a_2, a_3)\) has no positive solution over the integers. If \(h = 0\), we get solutions \((4, 0, 0)\) and \((1, 2, 0)\). In both cases, \(a_3 = 0\), which means there is no singular point of order 8, contradicting our assumption.

Using a similar argument, we identify \(\mathbb{Z}/4\) as a subgroups of \(\Gamma_1^1\). Nielsen’s Realization Theorem and the Riemann Hurwitz equation generate two possible solutions for the action of \(\mathbb{Z}/4\) on \(S_1^1\): \((a_1, a_2) = (4, 0)\) and \((a_1, a_2) = (1, 2)\), where \(a_1, a_2\) are the number of singular points of order 2 and 4 respectively. As \(a_2 \neq 0\), we only need consider \((a_1, a_2) = (1, 2)\). The \(\mathbb{Z}/4\) action on \(S_1\) has two fixed points of order 4 and one singular point of order 2.

The group \(\mathbb{Z}/2\) appears as a subgroup of \(\Gamma_1^1\) as it is a subgroup of \(\mathbb{Z}/4\). By a fixed point data argument that we do not include here, we determine that there is one conjugacy class of \(\mathbb{Z}/2\) in \(\Gamma_1^1\). The details of the fixed point data argument can be found in [16].

**The 3-torsion exponent:** We claim that there is no \(\mathbb{Z}/9\) in \(\Gamma_1^1\). We substitute \(g = 1\) and \(p = 3\) into the Riemann Hurwitz equation to get

\[
2g - 2 = p^2(2h - 2) + p^2(1 - 1/p)a_1 + p^2(1 - 1/p^2)a_2.
\]

There is one solution to the equation, \((a_1, a_2) = (3, 0)\), which is geometrically impossible because \(a_2 = 0\) and we must have at least one fixed point of order 9. However, there is a copy of \(\mathbb{Z}/3\) in \(\Gamma_1^1\). The corresponding solution to the Riemann Hurwitz equation generates an action on \(S_1\) has three fixed points corresponding to \(a_1 = 3\).

**The mixed torsion cases:** \(\mathbb{Z}/12\) is not a subgroup of \(\Gamma_1^1\). This follows because the Riemann Hurwitz equation,

\[
2g - 2 = 12(2h - 2) + 12(1 - 1/2)a_1 + 12(1 - 1/3)a_2
+ 12(1 - 1/4)a_3 + 12(1 - 1/6)a_4 + 12(1 - 1/12)a_5
\]

has no solution with \(a_5 \neq 0\), which is needed in order to have at least one fixed point of order 12. On the other hand, there is a copy of \(\mathbb{Z}/6\) in \(\Gamma_1^1\). The \(\mathbb{Z}/6\) action on \(S_1\) has one fixed point, one singular point of order 2 and one singular point of order 3 corresponding to the solution \((a_1, a_2, a_3) = (1, 1, 1)\).

We have now accounted for all the possible subgroups of finite order. By Corollary 5.4, the only possible infinite virtually cyclic groups are

\[
\{\mathbb{Z}, D_{\infty}, \mathbb{Z}/2, \mathbb{Z}/3 \times \mathbb{Z}, \mathbb{Z}/4 \times \mathbb{Z}, \mathbb{Z}/6 \times \mathbb{Z}, \mathbb{Z}/4 \times \mathbb{Z}/3, \mathbb{Z}/6 \times \mathbb{Z}/3, \mathbb{Z}/4 \times \mathbb{Z}/6, \mathbb{Z}/6 \times \mathbb{Z}/4 \times \mathbb{Z}, \mathbb{Z}/4 \times \mathbb{Z}/6 \times \mathbb{Z}, \mathbb{Z}/6 \times \mathbb{Z}/4 \times \mathbb{Z}, \mathbb{Z}/6 \times \mathbb{Z}/4 \times \mathbb{Z}/3, \mathbb{Z}/6 \times \mathbb{Z}/4 \times \mathbb{Z}/6\}\}
\]

From the calculations of the two exponent, we know
that there is only one conjugacy class of \( \mathbb{Z}/2 \) subgroups, and from the group presentation it is central. Therefore, there is no copy of \( D_\infty \) in \( \Gamma_1 \), hence no copies of \( \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4 \), and \( \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6 \) either. From results in [16], the normalizer of \( \mathbb{Z}/3 \) in \( \Gamma_1 \) is \( \mathbb{Z}/6 \). Thus, there is no copy of either \( \mathbb{Z}/3 \times \mathbb{Z}, \mathbb{Z}/6 \times \mathbb{Z}, \) or \( \mathbb{Z}/6 \times \mathbb{Z} \) in \( \Gamma_1 \). The claim follows. \( \square \)

Remark 5.7. We can actually do a better job of classifying the infinite virtual cyclic subgroups by using the fact that \( \Gamma_1 = \Gamma_1 \cong S\Gamma_0(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6 \). By a direct calculation, the virtually cyclic subgroups of \( S\Gamma_0(\mathbb{Z}) \) are \( \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6, \mathbb{Z}, \) and \( \mathbb{Z} \times \mathbb{Z} \). (For this last group, if \( s \) and \( t \) generate the copies of \( \mathbb{Z}/4 \) and \( \mathbb{Z}/6 \), then \( <s^2, st \gg \mathbb{Z}/2 \times \mathbb{Z} \) as \( s^2 = t^3 \).) All other direct and semidirect products should be trivial by results in Fine’s book. This implies that there are no copies of either \( \mathbb{Z}/4 \times \mathbb{Z} \) or \( \mathbb{Z}/4 \times \mathbb{Z} \) in \( S\Gamma_0(\mathbb{Z}) \). Using Sequence 3.2, these groups are not contained in \( \Gamma_1 \) either.

**Theorem 5.8.** Let \( \Gamma \) be a mapping class group of genus \( g = 1 \). Then for all \( i \leq 1 \), \( \tilde{K}_i(\mathbb{Z}\Gamma) = 0 \) with two exceptions: when \( \Gamma = \Gamma_0 \) or \( \Gamma_1 \), there is one non-vanishing \( K \)-group, namely \( K_{-1}(\mathbb{Z}\Gamma) = \mathbb{Z} \).

**Proof.** Of the list of all possible virtually cyclic subgroups, the only one with a non-vanishing lower algebraic \( K \)-group is \( \mathbb{Z}/6 \) \([17]\). Therefore Remark 7.1 implies that the only mapping class groups with possibly nonvanishing lower algebraic \( K \)-groups are \( \Gamma_0 \) and \( \Gamma_1 \). These two groups are isomorphic to \( S\Gamma_0(\mathbb{Z}) \cong \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4 \). As FIC holds for this group, an appropriate Mayer-Vietoris argument \([19]\) proves the claim. \( \square \)

6. OTHER EXAMPLES

The results in this paper apply to all groups which can be formed in stages like the mapping class groups. Prominent members of this family include the classical braid groups, braid groups of surfaces, and fiber-type arrangements.

**Definition 6.1.** Let \( M \) be a manifold without boundary. The configuration space of \( n \) ordered points of \( M \) is the space \( F(M, n) = \{(x_1, x_2, \ldots x_n) \in M^n | x_i \neq x_j \text{ if } i \neq j \} \).

When \( M = \mathbb{R}^2 \), the fundamental group of \( F(M, n) \) is Artin’s pure braid group on \( n \) strands. When \( M \) is a surface, the fundamental group of \( F(M, n) \) is called the pure braid group of \( M \) of \( n \) strands. Let \( Q_n \) denote \( m \) fixed distinct points in \( M \). There is a theorem due to Fadell and Neuwirth.
Theorem 6.2. [10] If $M$ is a (not necessarily compact) manifold without boundary, then there is a fibration of spaces $F(M \setminus Q_k, j - k) \to F(M, j) \to F(M, k)$.

It is known [8] that when $M$ is $\mathbb{R}^2$ or a compact surface of genus $g > 0$ the space $F(M \setminus Q_m, n)$ is a $K(\pi, 1)$. In these cases, the fibration in Theorem 6.2 leads to a short exact sequence of groups. We consider a special case of the fibration of aspherical spaces in Theorem 6.2,

$$F(M \setminus Q_{m+k}, 1) \to F(M \setminus Q_m, k + 1) \to F(M \setminus Q_m, k)$$

where $m$ may be equal to zero. Notice that $F(M, 1) = M$, so FIC holds for the fundamental group of $F(M, 1)$, whether $M$ is punctured or not. There is an associated sequence of homotopy groups

$$1 \to \pi_1(M \setminus Q_{m+k}) \to \pi_1(F(M \setminus Q_m, k + 1)) \to \pi_1(F(M \setminus Q_m, k)) \to 1.$$  

The group $\pi_1(M \setminus Q_{m+k})$ is a free group, and arguments in [1] show that this sequence satisfies the geometric realizability condition contained in the description of a strongly poly-free group. Therefore, Theorem 2.7 implies that FIC holds for $\pi_1(F(M \setminus Q_m, n))$ where $M$ is a surface as above.

This result is not new. A configuration space is the complement of what is known as a fiber-type arrangement, and Cohen proves that FIC holds for fiber-type arrangement groups in [7]. (A similar result is also proved in the Appendix of [11].) We can adapt Cohen’s techniques to Theorem 2.7, reproducing his result.

Theorem 2.7 occasionally offers a slight advantage over the techniques in [1] and [7]. By working with a group one extension at a time, it is easier to deal with groups that contain torsion. As examples, we apply the techniques in this paper to show that FIC holds for the configuration spaces for the 2-sphere and the real projective plane. Neither of these cases are considered in [1] and [7] because of the torsion they contain.

We first consider $F(S^2, k)$, the braid space of the 2-sphere. From Formula 1.7 in [5], the spaces in the fibration of Theorem 6.2 yield a short exact sequence

$$1 \to \pi_1(S^2 \setminus Q_{n-1}) \to \pi_1(F(S^2, n)) \to \pi_1(F(S^2, n - 1)) \to 1$$

for $n > 3$. It is known that $\pi_1(F(S^2, 3)) \cong \mathbb{Z}/2$. When $n = 4$, $\pi_1(F(S^2, 4))$ contains a free group (which is strongly poly-free) as a normal subgroup with $\mathbb{Z}/2$ quotient, so by Theorem 2.6, FIC holds for this case too. Theorem 2.7 and induction imply that FIC holds for $\pi_1(F(S^2, n))$ with $n > 4$. 
We case of $F(\mathbb{P}, k)$, the braid space on the real projective plane, is similar. By results of Van Buskirk in [23], there is a short exact sequence for $n \geq 3$,

$$1 \rightarrow \pi_1(\mathbb{P} \setminus Q_{n-1}) \rightarrow \pi_1(F(\mathbb{P}, n)) \rightarrow \pi_1(F(\mathbb{P}, n - 1)) \rightarrow 1,$$

where $\pi_1(\mathbb{P} \setminus Q_{n-1}) \cong F_{n-1}$ is a free group on $n - 1$ generators. Van Buskirk proves that $\pi_1(F(\mathbb{P}, 2)) \cong Q_8$, the quaternion group with eight elements. To start the induction, Theorem 2.6 implies that FIC holds for $\pi_1(F(\mathbb{P}, 3))$, as this group has a strongly poly-free normal subgroup with finite quotient. Induction and the short exact sequence of homotopy groups then implies that FIC holds for the rest of these braid spaces.

7. Appendix

We recall the Fibered Isomorphism Conjecture formulated in [12]. Let $\mathcal{S} : TOP \rightarrow SPECTRA$ be a covariant homotopy functor. Let $\mathcal{B}$ be the category of continuous surjective maps: objects in $\mathcal{B}$ are continuous maps $p : E \rightarrow B$, where $E, B$ are objects in $TOP$, and morphisms between $p_1 : E_1 \rightarrow P_1$ and $p_2 : E_2 \rightarrow P_2$ consist of continuous maps $f : E_1 \rightarrow E_2$ and $g_1 : B_1 \rightarrow B_2$ making the following diagram commute

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B_1 & \xrightarrow{g} & B_2.
\end{array}
$$

In this setup, Quinn [20] constructs a functor from $\mathcal{B}$ to $\Omega- SPECTRA$. The value of this spectrum at $p : E \rightarrow B$ is denoted by

$$\mathbb{H}(B; S(p)),$$

and has the property that its value at the object $E \rightarrow *$ is $S(E)$. The map of spectra $\mathbb{A}$ associated to

$$
\begin{array}{ccc}
E & \xrightarrow{id} & E \\
\downarrow{p} & & \downarrow{p} \\
B & \xrightarrow{g} & *.
\end{array}
$$

is known as Quinn assembly map.

Given a discrete group $\Gamma$, let $\mathcal{E}$ be a universal $\Gamma$-space for the family of virtually cyclic subgroups of $\Gamma$ [12, Appendix] and denote by $\mathcal{B}$ the orbit space $\mathcal{E}/\Gamma$. Let $X$ be any free and properly discontinuous $\Gamma$-space, and $p : X \times_{\Gamma} \mathcal{E} \rightarrow B$ be the map determined by the projection
onto $B$. The *Fibered Isomorphism Conjecture* (FIC) for $S$ and $\Gamma$ is the assertion that

$$A : \mathbb{H}(B; S(p)) \to S(X/\Gamma)$$

is a weak equivalence of spectra. This conjecture was made in [12, 1.7] for the functors $S = \mathcal{P}(), \mathcal{K}()$, and $L^{-\infty}$, the pseudoisotopy, algebraic $K$-theory and $L^{-\infty}$-theory functors. In this paper we mean FIC as FIC for the functor $S = \mathcal{P}()$.

**Remark 7.1.** It is known from [12, Lemma 1.4.2] that if a group $\Gamma$ is torsion-free and FIC holds for $\Gamma$, then $\tilde{K}_i(\mathbb{Z}\Gamma) = 0$ for $i \leq 1$. (Note that $Wh(\Gamma) = \tilde{K}_1(\mathbb{Z}\Gamma)$.) Moreover, the same conclusion is true when $\tilde{K}_i(\mathbb{Z}G) = 0$ for $i \leq 1$ and for all virtually cyclic subgroups $G$ of $\Gamma$.

**References**


LAFAYETTE COLLEGE, DEPARTMENT OF MATHEMATICS, EASTON, PA 18042
E-mail address: berkovee@lafayette.edu

INSTITUTO DE MATMÁTICAS, UNAM CAMPUS MORELIA, APARTADO POSTAL 61-3 (XANGARÍ), MORELIA, MICHOACÁN, MEXICO 58089
E-mail address: daniel@matmor.unam.mx

LAFAYETTE COLLEGE, DEPARTMENT OF MATHEMATICS, EASTON, PA 18042
E-mail address: luq@lafayette.edu