APPLICATIONS OF ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCES FOR MOTIVIC COBORDISMS

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ABSTRACT. We study applications of Atiyah-Hirzebruch spectral sequences for motivic cobordisms found by Hopkins and Morel.

1. INTRODUCTION

Let X be a smooth algebraic variety over a subfield k of the complex number field $\mathbb{C}$. To prove the Milnor conjecture, V.Voevodsky [Vo1,2] constructed the motivic cobordism theory $MGL^{*,*}(X)$ such that there are natural maps which commute the following diagram

$$\begin{array}{ccc} MGL^{*,*}(X) & \xrightarrow{t_{\mathbb{C}}} & MU^*(X(\mathbb{C})) \\ {\scriptstyle \rho^{*,*}}\downarrow & & {\scriptstyle \rho^{*}}\downarrow \\ H^{*,*}(X) & \xrightarrow{t_{\mathbb{C}}} & H^*(X(\mathbb{C})). \end{array}$$

Here $H^{*,*}(-)$ (resp. $MU^*(-)$, $H^*(-)$) is the motivic cohomology (resp. the complxes cobordism theory, the ordinary cohomology theory), $t_{\mathbb{C}}$ (resp. ρ^*) is the realization map (resp. the Thom map) and $X(\mathbb{C})$ is the complex manifold of rational $\mathbb{C}$ points of X.

To study $MGL^{*,*}(-)$, M.Hopkins and F.Morel ([Ho-Mo]) found the motivic Atiyah-Hirzebruch spectral sequence (AHss)

$$(1.1) \quad E(MGL)_2^{*,*,*} = H^{*,*}(X, MU^*) \Longrightarrow MGL^{*,*}(X).$$

Direct consequences of the existence of the motivic AHss are the isomophisms

$$(1.2) \quad MGL^{2*,*}(pt.) \cong MU^*, \quad \textit{and} \quad MGL^{2*,*}(X) \otimes_{MU^*} \mathbb{Z} \cong CH^*(X)$$

where $CH^*(X)$ is the classical Chow ring of X. There are many other applications.

Given a regular sequence $S = (a_1, a_2, ...)$,$a_i \in MU^*$, there is the generalized (topological) cohomology theory $MU(S)^*(-)$ ([Sh-Ya]) such that the coefficient ring is isomorphic to $MU(S)^* = MU^*/(Ideal(S))$. In particular, $MU(x_1, x_2, ...) \cong H\mathbb{Z}$ where $MU^* = \mathbb{Z}[x_1, x_2, ...]$ and $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum.

Similarly we can construct the algebraic cohomology theory $AMU(S)^{*,*}(-)$ such that $t_{\mathbb{C}}(AMU(S)) \cong MU(S)$. The crucial point of the proof by Hopkins-Morel of the existence of the motivic AHss is also the isomorphism $AMU(x_1, x_2, ...) = AH\mathbb{Z} \cong H_{\mathbb{Z}}$: the spectrum representing the motivic cohomology.

We give a proof of the fact that $AMU(p, x_1, ...) \cong H_{\mathbb{Z}/p}$ because the proof is quite easy for this case and most computations in this paper are given only for

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$mod(p)$ theories. However we use the $\mathbb{A}^1$-homotopy Whitehead theorem and use the fact that the motivic Steenrod algebra $A_p^{*,*}$ is multiplicatively generated by $H^{*,*}(pt;\mathbb{Z}/p)$, reduced powers P^i and the Milnor operations Q_j. The last fact is proved by V.Voevodsky and G.Powell [Po], while it is not published yet.

¿From this fact we get the Sullivan-Baas exact sequences and AHss for many motivic cohomology theories, e.g., $ABP, AP(n), ABP\langle n\rangle, Ak(n)$. The differentials of AHss for these theories are related to the Milnor opration, in fact,

$$(1.3)\quad d_{2p^n-1}(x) = v_n \otimes Q_n(x) \quad mod(Q_0,...,Q_{n-1})$$

in AHss for $AP(n)^{*,*}(-)$ theory where $BP^* \cong \mathbb{Z}_{(p)}[v_1,v_2,...]$ as usual.

The following are applications of AHss for motivic theories.

Recall that $BP\langle n\rangle^*(X)$ is the theory with the coefficient $BP\langle n\rangle^* = \mathbb{Z}_{(p)}[v_1,...,v_n]$. It is proved that if $ABP\langle n\rangle^*(X)$ is $(p,v_1,...,v_n)$-torsion, then $H^{*,*}(X;\mathbb{Z}/p)$ is a $\Lambda(Q_0,...,Q_n)$-free module. For a nonzero element $a=(a_0,...,a_n)$ in the Milnor K-theory $K_{n+1}^M(k)$, let $\tilde{C}(Q_a)$ be the reduced $\check{C}ech$ complex defined from the norm quadric Q_a of dimesnsion 2^n-1 (see [Vo1,2] for details). Then we see that $H^{*,*}(\tilde{C}(Q_a);\mathbb{Z}/2)$ is $\Lambda(Q_0,...,Q_n)$-free. This gives a little different proof of some parts in the proof of Milnor conjecture by Voevodsky. Moreover we show that

$$(1.4)\quad H^{*,*}(\tilde{C}(Q_a);\mathbb{Z}/2) \cong K_*^M(k)/(Ker(a)) \otimes \Lambda(Q_0,...,Q_n) \otimes \mathbb{Z}/2[\delta_a^2]\{a\tau^{-1}\}$$

where $deg(\delta_a^2) = (2^{n+2}-2, 2^{n+1}-2)$ and $deg(a\tau^{-1}) = (n+1,n)$.

Hu and Kriz ([Hu-Kr1,2]) studied the Real cobordism theories. In particular, they computed the coefficient of the Real BP-theory $BP\mathbb{R}^{*,*}(pt)$. By the Real realization map $t_{\mathbb{R}}$, the Real theories are related to the motivic theories over $k=\mathbb{R}$. We see that $ABP/2^{*,*}(Spec(\mathbb{R}))$ is a $BP/2^*$-submodule of $BP/2\mathbb{R}^{*,*}(pt)$ and

$$(1.5)\ grABP^{*,*}/2(Spec(\mathbb{R})) \cong BP^*[\rho]\{1, v_n t_2^{2^{n-1}\ell} | \ell, n \geq 1\}/(2, v_n\rho^{2^{n+1}-1}) \otimes \Lambda(t_0)$$

where $deg(t_i) = (0,2^i)$, $deg(\rho) = (1,1)$.

Recall that $P(n)^*(-)$ (resp. $K(n)^*(-)$) is the theory with $P(1)^* \cong \mathbb{Z}/p[v_n,...]$ (resp.$K(n)^* \cong \mathbb{Z}/p[v_n,v_n^{-1}]$). When $k=\mathbb{C}$, we see that $AP(n)^{*,*}(-)$ and $AK(n)^{*,*}(-)$ work very well. In particular,

$$(1.6)\quad AP(n)^{*,*}(AP(n)\wedge AP(n)) \cong P(n)^*(P(n)\wedge P(n)) \otimes \mathbb{Z}/p[\tau]$$

where $deg(\tau) = (0,1)$. Then we prove that $AP(n)^{*,*}(-)$ has a good product and holds the Conner-Floyd type theorem, i.e., $AK(n)^{*,*}(-) \cong K(n)^* \otimes_{K(n)^*} AP(n)^{*,*}(-)$ ([Ya1],[Wu]).

Let $\Omega_{BP}^*(-)$ be the BP-version of the algebraic cobordism $\Omega^*(-)$ defined by Morel and Levine ([Mo-Le1,2]). Let G be an algebraic group over $\mathbb{C}$ and BG be its classifying space [To1,2]. We note that

$$(1.7)\quad ABP^{2*,*}(BG) \cong \Omega_{BP}^*(BG) \cong BP^*(BG)$$

hold for all groups with no p-torsion in $H^*(BG)$ or finite abeilan groups. However for nonabeian p-groups, we give only weeker results.

Next consider another type of groups. Let G be a simply connected Lie group and $G_{\mathbb{C}}$ be the corresponding reductive algebraic group. Let us write $H^*(G;\mathbb{Z}/p) \cong P(y) \otimes \Lambda(x_1,...,x_\ell)$ where $P(y)$ is a truncated polynomial generated by even dimensional generators. Let $I=(p,v_1,..)$ be the (invariant prime) ideal in BP^*. We

also note that we have the isomorphism of BP^*-modules

$$(1.8)\quad ABP^{2*,*}(G_{\mathbb{C}})/I^2 \cong \Omega^*_{BP}(G_{\mathbb{C}})/I^2 \cong BP^*\otimes P(y)/(I_2^2, \sum_j v_jQ_jx_i|1\le \imath \le \ell).$$

The last isomorphism is proved in [Ya5], while the first isomorphism is immedite consequence from the existence of AHss (1.2).

Section 2 (resp. 8) is a short introduction to the motivic cohomology therory (resp. algebraic cobordism). In §3, we define $Ah^{*,*}(-)$ theory for sertain toplogical theory $h^*(-)$ and prove $AH\mathbb{Z}/p \cong H_{\mathbb{Z}/p}$, and give AHss for these theories. In §4, we note the relation of Q_i and the differential of AHss. In §5, we study $\tilde{C}(Q_a)$ and the Rost motives. In §6, $ABP^{*,*}/2(Spec(\mathbb{R}))$ is studied. In §7, we study $AP(n)^{*,*}(-)$ theories for $k=\mathbb{C}$. Section 9 (resp. 10) treats the $ABP^{*,*}(-)$ for classifying spaces BG for algebraic groups over $\mathbb{C}$ (resp. the reductive groups $G_{\mathbb{C}}$ for simply connected Lie groups).

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2. MOTIVIC COHOMOLOGY

We use some category *Spc* of (algebraic) spaces, defined by Voevodsky, where schemes A, their quotients A_1/A_2 and $colim(A_\alpha)$ are all contained ([Vo2],[Mo-Vo]). Here schemes are defined over a field k with $ch(k)=0$. The motivic cohomology is the double indexed cohomology $H^{*,*}(X)$ defined by Suslin and Voevodsky directely related with the Chow ring and the Milnor K-theory. Indeed, for a smooth scheme X, $H^{2n,n}(X)\cong CH^n(X)$, the classical Chow group of codim n cycles on X modulo rational equivalence. The cohomology $H^{n,n}(Spec(k))\cong K_n^M(k)$; the Milnor K-group for the field k.

Since *Spc* contains *colimit*, we can consider the infinite projective space $\mathbb{P}^\infty = B\mathbb{G}_m$ and the infinite Lens space $colim_n(\mathbb{A}^n-\{0\}/\mathbb{Z}/p) = L_p^\infty = B\mathbb{Z}/p$. The Chow rings of these spaces are given [To1] by

$$(2.1)\quad CH^*(\mathbb{P}^\infty)\cong H^{2*,*}(\mathbb{P}^\infty)\cong \mathbb{Z}[y],\quad CH^*(B\mathbb{Z}/p)\cong H^{2*,*}(B\mathbb{Z}/p)\cong \mathbb{Z}[y]/(py)$$

with $deg(y)=(2,1)$.

The Milnor K-theory is the graded ring $\oplus_n K_n^M(k)$ defined by $K_n^M(k)=(k^*)^{\otimes n}/J$ where the ideal J is generated by elements $a\otimes(1-a)$ for $a\in k^*-\{1\}$. Here the addition of k^* is given by the multiplication in the field k. Hence $K_0^M(k)=\mathbb{Z}$ and $K_1^M(k)=k^*$. Hilbert's theorem 90, which is essentialy said that the Galois cohomology $H^1(G(k_s/k);k_s^*)=0$, implies the isomorphism $K_1^M(k)/p\cong k^*/(k^*)^p\cong H^1(G(k_s/k);\mathbb{Z}/p)$ for $1/p\in k$. Similarly we can define a map (the norm residue map) for any extension F of k of finite type

$$(2.2)\qquad K_n^M(F)/p \to H^n(G(F_s/F);\mu_p^{\otimes n})$$

where $\mu_p^{\otimes n}$ is the discrete $G(F_s/F)$-module of n-th tensor power of the group of p-roots of 1. The Bloch-Kato conjecture is that this map is an isomorphism for all field k and the Milnor conjecture is its $p=2$ case. This conjecture is solved when $n=2$ by Merkurjev-Susulin [Me-Su], and for $p=2$ by Voevodsky [Vo1].

Notice that $H^n(G(k_s/k);\mu_p^{\otimes n})\cong H^n_{et}(Spec(k),\mu_p^{\otimes n})$ the étale cohomology of the point. The étale cohomology $H^*_{et}(X;\mathbb{Z}/p)$ has the following properties. If k

contains a primitive p-th root of 1, then there is the additive isomorphism

$$(2.3) \quad H^m_{et}(X, \mu_p^{\otimes n}) \cong H^m_{et}(X; \mathbb{Z}/p).$$

For smooth X over $k = \mathbb{C}$,

$$(2.4) \quad H^m_{et}(X; \mathbb{Z}/p^N) \cong H^m(X(\mathbb{C}); \mathbb{Z}/p^N) \quad for\ all\ N \geq 1.$$

The last cohomology is the usual mod p ordinary cohomology of $\mathbb{C}$-rational point of X. Of course $H^*_{et}(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p$. It is also known that $H^*_{et}(Spec(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]$ with $deg(\rho) = 1$ for the real number field $\mathbb{R}$.

We recall the Lichtenbaum motivic cohomology ([Vo1,2]). Lichtenbaum defined the similar cohomology $H^{*,*}_L(X; R)$ by using the étale topology, while $H^{*,*}(X; R)$ is defined by using Nisnevich topology. Since Nisnevich covers are some restricted étale covers, there is the natural map $H^{*,*}(X; R) \to H^{*,*}_L(X; R)$. We say that the condition $B(n, p)$ holds if

$$(2.5) \quad H^{m,n}(X; Z_{(p)}) \cong H^{m,n}_L(X; Z_{(p)}) \quad for\ all\ m \leq n + 1$$

and all smooth X. The Beilinson-Lichtenbaum conjecture is that $B(n, p)$ holds for all n, p. It is proved that the $B(n, p)$ condition is equivalent the Bloch-Kato conjeture for degree n and prime p. Hence $B(n, p)$ holds for $n \leq 2$ or $p = 2$. Moreover Suslin-Voevodsky proves

$$(2.6) \quad H^{m,n}_L(X; \mathbb{Z}/p) \cong H^m_{et}(X; \mu_p^{\otimes n}).$$

When $k \subset \mathbb{C}$, there are maps (realization maps)

$$t^{m,n}_{\mathbb{C}} : H^{m,n}(X; \mathbb{Z}/p) \to H^m(X(\mathbb{C}); \mathbb{Z}/p)$$

which sum up $t^{*,*}_{\mathbb{C}} = \oplus_{m,n} t^{m,n}_{\mathbb{C}}$ the natural ring homomorphism. Suppose that $B(n, p)$ condition holds. When $k = \mathbb{C}$, by isomorphisms (2.3)-(2.6), we have

$$(2.6)' \quad H^{n,n}(X; \mathbb{Z}/p) \cong H^n(X(\mathbb{C}); \mathbb{Z}/p).$$

This isomorphism also represents the realization map $t^{n,n}_{\mathbb{C}}$.

Now we compute $H^{*,*}(pt; \mathbb{Z}/p) = H^{*,*}(Spec(k); \mathbb{Z}/p)$. Define the weight (resp. difference) $w(x) = 2n - m$ (resp. $d(x) = m - n$) for $deg(x) = (m, n)$. For a smooth X, if $H^{m,n}(X; \mathbb{Z}/p) \not\cong 0$, then it is known ([Vo1,2]) that

$$(2.7) \quad w(x) \geq 0, \quad d(x) \leq dim(X).$$

We also note $w(x) = 0$ if $x \in CH^*(X)$ and $w(xy) = w(x) + w(y)$, $d(xy) = d(x) + d(y)$.

Hereafter this paper,we assume that k contains a primitive p-th root of 1 and $B(n, p)$ holds for all n but $X = Spec(k)$. Then

$$H^{m,n}(pt; \mathbb{Z}/p) \cong H^m_{et}(pt; \mu_p^{\otimes n}) \cong H^m_{et}(pt; \mathbb{Z}/p) \qquad if\ m \leq n$$

and $H^{m,n}(pt; \mathbb{Z}/p) \cong 0$ for $m > n$. Let $\tau \in H^{0,1}(pt; \mathbb{Z}/p)$ be the element corresponding a generator of $H^0_{et}(Spec(k); \mu_p) \cong H^0_{et}(Spec(k); \mathbb{Z}/p) \cong \mathbb{Z}/p$. Then we get the isomorphism

$$H^{*,*}(Spec(k); \mathbb{Z}/p) \cong H^*_{et}(Spec(k); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since $\tau : H^m_{et}(pt; \mu_p^{\otimes n}) \cong H^m_{et}(pt; \mu_p^{\otimes (n+1)})$. For examples,

$$(2.8) \quad H^{*,*}(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau], \quad H^{*,*}(Spec(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau, \rho].$$

Next we compute cohomology of $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$. Let $R=\mathbb{Z}$ or $\mathbb{Z}/p$. For the projective space $\mathbb{P}^n$, there is an isomorphism

$$(2.9)\qquad H^{*,*}(X\times\mathbb{P}^n/\mathbb{P}^{n-1};R)\cong H^{*,*}(X;R)\{1,y'\}$$

with $deg(y')=(2n,n)$ and $t_{\mathbb{C}}(y')\neq 0$ for $k\subset\mathbb{C}$. For any (algebraic) map $f:X\to Y$ in the category Spc, we can construct the cofiber sequence

$$X\to Y\to cone(f)=Y/X$$

which induces the long exact sequence (Voevodsky [Vo2],[Mo-Vo])

$$(2.10)\quad H^{*,*}(X;R)\leftarrow H^{*,*}(Y;R)\leftarrow H^{*,*}(Y/X:R)\leftarrow H^{*-1,*}(X;R).$$

In particular, we get the Mayer-Vietoris, Gysin and blow up long exact sequences.

By the cofiber sequence $\mathbb{P}^{n-1}\to\mathbb{P}^n\to\mathbb{P}^n/\mathbb{P}^{n-1}$ and (2.9), we can inductively see that

$$(2.11)\quad H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\cong H^{*,*}(pt;\mathbb{Z}/p)\otimes\mathbb{Z}/p[y]/(y^{n+1})\quad with\quad deg(y)=(2,1).$$

For the Lens spaces, we also have the cofiber sequence in the $\mathbb{A}^1$-stable category $SHot$ (see Section 3 bellow)

$$(2.12)\quad L_p^n\to\mathbb{P}^n\xrightarrow{\times p}\mathbb{P}^n.$$

Thus we get the additive isomorphism $H^{*,*}(L_p^n;\mathbb{Z}/p)\cong H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\{1,x\}$. This induces the ring isomorphism for $p=odd$ ([Vo3])

$$(2.13)\quad H^{*,*}(L_p^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y]/(y^{n+1})\otimes\Lambda(x)\otimes H^{*,*}(pt;\mathbb{Z}/p)$$

with $deg(x)=(1,1)$. However note that when $p=2$, we see $x^2=y\tau+x\rho$ [Vo3] where $\rho\in H^{1,1}(pt;\mathbb{Z}/p)\cong k^*/k^{2*}$ represents -1. (Hence $\rho=0$ when $\sqrt{-1}\in k^*$.) This is proved by the wellknown fact $\{a,a\}=\{a,-1\}$ in the Milnor K-theroy $K_2^M(k)$.

Let us say that a space X satisfies the Kunneth formula for a space Y if

$$H^{*,*}(X\times Y;\mathbb{Z}/p)\cong H^{*,*}(X;\mathbb{Z}/p)\otimes_{H^{*,*}(pt;\mathbb{Z}/p)}H^{*,*}(Y;\mathbb{Z}/p).$$

By the above cofiber sequences, we can easily see that $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$ satify the Kunneth formula for all spaces. In particular, we have the ring isomorhisms

$$(2.14)\quad H^{*,*}((\mathbb{P}^\infty)^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes H^{*,*}(pt;\mathbb{Z}/p)$$

$$(2.15)\quad H^{*,*}((B\mathbb{Z}/p)^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes\Lambda(x_1,...,x_n)\otimes H^{*,*}(pt;\mathbb{Z}/p)$$

(when $p=2$, $x_i^2=y_i\tau+x_i\rho$).

This fact is used to defined the reduced power operation P^i. Indeed, we have (the Bockstein, the reduced powers) operations

$$(2.16)\qquad \beta:H^{*,*}(X;\mathbb{Z}/p)\to H^{*+1,*}(X;\mathbb{Z}/p)$$

$$P^i:H^{*,*}(X;\mathbb{Z}/p)\to H^{*+2(p-1)i,*+(p-1)i}(X;\mathbb{Z}/p),$$

which commutes with the realization map $t_{\mathbb{C}}$ when $k\subset\mathbb{C}$. Moreover we have the Milnor primitive operation

$$Q_i:H^{*,*}(X;\mathbb{Z}/p)\to H^{*+2p^i-1,*+p^i-1}(X;\mathbb{Z}/p),$$

such that $Q_i=[Q_{i-1},P^{p^{i-1}}]\ modulo(\rho)$ (see [Vo3] for details). Note that $w(P^i)=0,w(Q_i)=-1,w(\tau)=2$, and $d(P^i)=i(p-1),d(Q_i)=2^i,d(\tau)=-1$.

V.Voevodsky and G.Powell [Po] also showed that the mod p motivic Steenrod algebra $A_p^{*,*}$ is generated as an $H^{*,*}(pt,\mathbb{Z}/p)$-module by the product of P^i and β, in particular

$$(2.18) \quad A_p^{*,*} \cong H^{*,*}(pt;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_0, Q_1, ...)$$

where RP is the $\mathbb{Z}/p$-module generated by products of reduced powers.

We can see ([Ho-Kr 2])

$$(2.19) \quad H^{*,*}(BGL_n;\mathbb{Z}) \cong \mathbb{Z}[c_1, ..., c_n] \otimes H^{*,*}(pt;\mathbb{Z})$$

where the Chern class c_i with $deg(c_i) = (2i, i)$ are identified with the elementary symmetric polynomial in $H^{*,*}((\mathbb{P}^\infty)^n;\mathbb{Z})$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG;\mathbb{Z})$ for each representation $\rho : G \to GL_n$.

3. GENERALIZED MOTIVIC COHOMOLOGIES

¿From the category Spc, Morel and Voevodsky constructs ([Vo1,2],[Mo-Vo]) the ($\mathbb{A}^1$, algebraic) homotopy category Hot and stable homotopy category $SHot$. There are two different types of spheres in Spc

$$(3.1) \qquad S_s^1 = \mathbb{A}^1/\{0,1\} \quad and \quad S_t^1 = \mathbb{A}^1 - \{0\}.$$

The Tate object is $T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{P}^1 \cong S_t^1 \wedge S_s^1$ in Hot. The category $SHot$ is defined by the T as the suspension, e.g., $E = \{E_i\}$, $E_i \in Spt$ is a spectrum if there are the maps $T \wedge E_i \to E_{i+1}$.

Let Σ_T^∞ be the functor from Spc to T-spectra by X to $\{T^i \wedge X\}$. If E is a T-spectrum, then the motivic (generalized) cohomology $E^{*,*}(-)$ is defined by

$$(3.2) \qquad E^{m,n}(X) = Hom_{SHot}(\Sigma_T^\infty(X), S^{m,n} \wedge E)$$

where $S^{m,n} = S_s^{m-n} \wedge S_t^n$ and $Hom_{SHot}(-,-)$ are homomorphisms defined on $SHot$.

The spectrum for the ordinary motivic cohomology is defined as following. (See [Vo 1] for details.) Let $L(X;R)$ for $R = \mathbb{Z}$ or $\mathbb{Z}/p$ be the presheaf sending a connected U to the free R-module genrated by the set of all closed irreducible $W \subset U \times X$ such that the projection $W \to U$ are finite and surjective. The Eilenberg-MacLane spectrum is defined as

$$(3.3) \quad K(R(n), 2n) = L(\mathbb{A}^n; R)/L(\mathbb{A}^n - \{0\}; R).$$

We also write the specrum $\{K(R(n), 2n)\}$ by H_R.

Let BGL denote the infinite Grassmanian, the union over N of $GL_N(\infty)$.The corresponding generalized cohomology theory is the algebraic K-theory. The motivic cobordism theory $MGL^{*,*}(-)$ is the generalized cohomology theory defined by the Thom spectrum $MGL = \{Th(E_n \to GL_n)\}_n$ with identifying $Th(E_n \oplus O) \cong T \wedge Th(E_n)$ and $E_n \oplus O \to E_{n+1}$ for the n-dimensional universal bundle E_n and the trivial line bundle O.

Recall that $H^{*,*}(BGL;\mathbb{Z}) \cong H^{*,*}(pt;\mathbb{Z}) \otimes \mathbb{Z}/p[c_1, ...]$. It is known that as mod p-Steenrod algebra A_p^*-modules

$$\mathbb{Z}/p[c_1, ...] \cong RP \otimes \mathbb{Z}/p[m_i | i \neq p^j - 1] \quad with \ |m_i| = 2i$$

where RP is the subalgebrta of A_p^* generated by reduced powers. By Thom isomorphism $H^{*,*}(MGL;\mathbb{Z}/p) \cong H^{*,*}(BGL;\mathbb{Z}/p)$, we have the mod p -motivic Steenrod

algebra $A_p^{*,*}$-modules isomorphism (see Borghesi [Bo] also)

$$H^{*,*}(MGL;\mathbb{Z}/p) \cong H^{*,*}(pt;\mathbb{Z}/p) \otimes RP \otimes \mathbb{Z}/p[m_i | i \neq p^j - 1].$$

where $deg(m_i) = (2i, i)$. Moreover we have the isomorphism of the integral coefficient case

$$H^{*,*}(MGL;\mathbb{Z}) \cong H^{*,*}(pt;\mathbb{Z}) \otimes \tilde{RP} \otimes \mathbb{Z}[m_i | i \neq p^j - 1] \tag{3.4}$$

where $\tilde{RP} = \{\alpha = (\alpha_1, ...) | \alpha_i \geq 0\}$ with $deg(\alpha) = (\sum_i 2(p^i - 1)\alpha_i, \sum_i (p^i - 1)\alpha_i)$ so that $\tilde{RP}/p = RP$.

Let us write by AMU the spectrum $MGL_{(p)}$ representing the motivic cobordism theory, i.e., $MGL^{*,*}(-)_{(p)} = AMU^{*,*}(-)$. Given a regular sequence $S_n = (a_1, ..., a_n)$ with $a_i \in MU^*_{(p)}$, we can inductively construct the AMU-module spectrum by

$$T^{-1/2|a_i|} \wedge AMU(S_{i-1}) \xrightarrow{\times a_i} AMU(S_{i-1}) \to AMU(S_i). \tag{3.5}$$

Here we can easily see that $AMU(S_n)$ is an AMU-module spectrum, while it seems not easy to see AMU-ring spectrum. It is also immediate that $t_{\mathbb{C}}(AMU(S_n)) \cong MU(S_n)$ with $MU(S_n)^* = MU^*/(Ideal(S_n))$.

Recall $MU^* \cong \mathbb{Z}[x_1, ...]$ with $|x_i| = -2i$ and $BP^* = \mathbb{Z}_{(p)}[v_1, ...]$ with identifying $v_i = x_{p^i - 1}$. We can construct spectra

$$ABP = AMU(x_i | i \neq p^j - 1),\ AP(n),\ Ak(n),\ AK(n),\ AH\mathbb{Z},\ AH\mathbb{Z}/p$$

so that $t_{\mathbb{C}}(Ah) \cong h$ for $h = BP, P(n), ...$ Here $P(n)^* = BP^*/(p, ..., v_{n-1})$ and $k(n)^* = \mathbb{Z}/p[v_n] \cong BP^*/(p, ..\hat{v}_n, ...)$.

For $i \neq p^j - 1$, the induced map $(\times x_i)^* : H^{*,*}(AMU;\mathbb{Z}/p) \to H^{*,*}(AMU;\mathbb{Z}/p)$ is the map multiplying by m_i . Indeed $(x_i)^* = m_i$ in $H^*(MU;\mathbb{Z})$. Of course $w((x_i)^*(y)) = 0$ for $y \in \tilde{RP} \otimes \mathbb{Z}[m_i]$. This fact implies $(x_i)^* = m_i$ also in $H^{*,*}(AMU;\mathbb{Z})$ since $w(a) > 0$ if $0 \neq a \in H^{*,*}(pt;\mathbb{Z}))$ for $\{*,*\} \neq \{0,0\}$. Thus we know that

$$H^{*,*}(AMU(x_i);\mathbb{Z}) \cong H^{*,*}(AMU;\mathbb{Z})/(m_i).$$

Considering this argument for all $m_i, i \neq p^j - 1$, we have

$$H^{*,*}(ABP;\mathbb{Z}) \cong H^{*,*}(pt;\mathbb{Z}) \otimes \tilde{RP}. \tag{3.6}$$

Lemma 3.1. *For $n \geq 1$, we have*

$$H^{*,*}(AP(n);\mathbb{Z}/p) \cong H^{*,*}(pt;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_0, .., Q_{n-1}).$$

Proof. By the cofibration

$$T^{p^n - 1} \wedge AP(n) \xrightarrow{v_n} AP(n) \to AP(n+1) \tag{3.7}$$

we have the long exact sequence

$$\xleftarrow{\delta_n} H^{*-2p^n+1, *-p^i+1}(AP(n);\mathbb{Z}/p) \xleftarrow{v_n^*} H^{*,*}(AP(n);\mathbb{Z}/p) \leftarrow H^{*,*}(AP(n+1);\mathbb{Z}/p).$$

By dimensional reason, $v_n^* \iota = 0$ for $\iota \in H^{0,0}(AP(n);\mathbb{Z}/p) \cong \mathbb{Z}/p$. This induces that $v_n^*(x) = 0$ for all $x \in H^{*,*}(AP(n);\mathbb{Z}/p)$ since x is represented as

$$x = \theta\iota : AP(n) \xrightarrow{\ \iota\ } H_{\mathbb{Z}/p} \xrightarrow{\ \theta\ } S^{*,*} \wedge H_{\mathbb{Z}/p}$$

for $\theta \in H^{*,*}(pt.;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_0, ..., Q_{n-1})$ by inductive assumption. Hence we get

$$H^{*,*}(AP(n+1);\mathbb{Z}/p) \cong H^{*,*}(AP(n);\mathbb{Z}/p)\{1, \delta_n\}.$$

Here we know $t_{\mathbb{C}}(\delta_n)=Q_n$ in $H^{*,*}(P(n+1);\mathbb{Z}/p)$ ([Sh-Y],[Ya1],[Wu]). Hence

$$\delta_n = Q_n + \sum_{IJ} a_{IJ}P^IQ_J \qquad \textit{with } |J|>0,\ a_{IJ}\in H^{plus,*}(pt;\mathbb{Z}/p).$$

Thus we can inductively prove the lemma. □

In particular when $n=\infty$, we get

$$H^{*,*}(AH\mathbb{Z}/p;\mathbb{Z}/p)\cong H^{*,*}(H_{\mathbb{Z}/p};\mathbb{Z}/p)\cong H^{*,*}(pt;\mathbb{Z}/p)\otimes RP\otimes\Lambda(Q_0,...).$$

By Morel [Mo1], it is known that $\mathbb{A}^1$-homotopy theory holds the Hurwitz theorem. This induces the Whitehead theorem, namely, if there is a map of connected spectra

$$f:X\to Y \qquad \textit{such that} \quad H^{*,*}(X)\cong H^{*,*}(Y),$$

then X is equivalent to Y in $SHot$.

Theorem 3.2. *In the stable homotopy category $SHot$, we get $H_{\mathbb{Z}/p}\cong AH\mathbb{Z}/p$, e.g., $AH\mathbb{Z}/p^{*,*}(X)\cong H^{*,*}(X;\mathbb{Z}/p)$.*

Proof. The cofiber sequence $AH\mathbb{Z}\xrightarrow{p}AH\mathbb{Z}\to AH\mathbb{Z}/p$ induces the long exact sequence

$$\longleftarrow H^{*,*}(AH\mathbb{Z})\xleftarrow{p^*}H^{*,*}(AH\mathbb{Z})\longleftarrow H^{*,*}(AH\mathbb{Z}/p).$$

Here $p^*=p$ since the both elements $p^*(x)$ and px are represented by the stable map

$$S^{i,j}\wedge AH\mathbb{Z}\wedge S^0 \xrightarrow{x\wedge p} H_{\mathbb{Z}}\wedge S^0 \qquad \textit{for} \ \ x:S^{i,j}\wedge AH\mathbb{Z}\to H_{\mathbb{Z}}.$$

Hence all elements in $H^{*,*}(AH\mathbb{Z}/p)$ are p^2-torsion (no infinitely divisible element). Next consider the exact sequence

$$\longrightarrow H^{*,*}(AH\mathbb{Z}/p)\xrightarrow{p}H^{*,*}(AH\mathbb{Z}/p)\longrightarrow H^{*,*}(AH\mathbb{Z}/p;\mathbb{Z}/p).$$

Here $H^{*,*}(AH\mathbb{Z}/p;\mathbb{Z}/p)$ is $\Lambda(Q_0)$-free and hence all elements in $H^{*,*}(AH\mathbb{Z}/p)$ are just p-torsion. Thus we have

$$(3.8)\quad H^{*,*}(AH\mathbb{Z}/p)\cong RP\otimes\Lambda(Q_1,Q_2,...)\{Q_0\}.$$

The same fact holds for $H_{\mathbb{Z}/p}$ and we have the isomorphism

$$(3.9)\quad H^{*,*}(H_{\mathbb{Z}/p})\cong H^{*,*}(AH\mathbb{Z}/p).$$

By the Whitehead theorem for $\mathbb{A}^1$-stable homotopy category, we get the theorem. □

Direct application of this fact is that from the cofiber sequnce

$$T^{p^n-1}\wedge Ak(n)\xrightarrow{v_n}Ak(n)\longrightarrow AH\mathbb{Z}/p=H_{\mathbb{Z}/p}$$

we have the Bockstein spectral sequence

$$E_2^{*,*,*}=H^{*,*}(X;\mathbb{Z}/p)\otimes k(n)^*\Longrightarrow Ak(n)^{*,*}(X).$$

More strongly, Hopkins-Morel proves the following theorems.

Theorem 3.3. *([Ho-Mo]) $AH\mathbb{Z}\cong H_{\mathbb{Z}}$, namely, $AH\mathbb{Z}^{*,*}(X)\cong H^{*,*}(X,\mathbb{Z})$; the motivic cohomology.*

Theorem 3.4. *([Ho-Mo]) Let $Ah = AMU(S_n)$ for some regular sequence $S_n = (a_1, ..), a_i \in MU^*$. Then there is the Atiyah-Hirzebruch spectral sequence*

$$E(Ah)_2^{(m,n,2n')} = H^{m,n}(X; h^{2n'}) \Longrightarrow Ah^{m+2n',n+n'}(X)$$

with the differential $\quad d_{2r+1} : E_{2r+1}^{(m,n,2n')} \to E_{2r+1}^{(m+2r+1,n+r,2n'-2r)}$.

Let $K^{*,*}(X)$ be the algebraic K-theory. Let $\tilde{K}(1)^* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$ (which is the coefficient ring of the algebraic integral Morava K-theory $\tilde{K}(1)^*(-)$).

Theorem 3.5. *([Ho-Mo])*

$$K^{*,*}(X) \otimes \tilde{K}(1)^* \cong \tilde{K}(1)^* \otimes_{MU^*_{(p)}} MGL^{*,*}(X)_{(p)}.$$

Corollary 3.6. *If X is smooth, then $Ah^{2*+j,*}(X) \cong 0$ for $j > 0$,i.e., $w(x) \geq 0$ for all nonzero element in $x \in Ah^{*,*}(X)$.*

Proof. Since $H^{2*+j,*}(X) = 0$, we see $E_r^{2*+j,*,2k} \cong 0$ for all $j > 0$.
Hence $Ah^{2*+2k+j,*+k}(X) = 0$. □

Corollary 3.7. $Ah^{2*,*}(pt) \cong h^{2*}$.

Proof. Since $H^{2*,*}(pt) = H^{0,0}(pt) \cong \mathbb{Z}$, each element in $E_2^{2*,*,2*'} \cong E_2^{0,0,2*'} \cong h^{2*'}$ is permanent cycle in the spectral sequence. □

By dimensinal reason of the differential of AHss, we have ;

Corollary 3.8. *If X is smooth, then $AMU(S_n)^{2*,*}(X) \otimes_{MU^*_{(p)}} \mathbb{Z}/p \cong CH^*(X)/p$. Moreover if $MU^*_{(p)}/(S_n)$ is $\mathbb{Z}_{(p)}$-free, then $AMU(S_n)^{2*,*}(X) \otimes_{MU^*_{(p)}} \mathbb{Z}_{(p)} \cong CH^*(X)$.*

For example, we see

$$Ah^{*,*} \supset h^* \otimes (\oplus_{i=0}^{2p-1} K_i^M(k))$$

where $K_i^M(k)$ is the Milnor K-group of degree i for the field k. In particular, if h^* is a $\mathbb{Z}/p$-module and $k = \mathbb{C}$, then we have the isomorphism

$$Ah^{*,*}(Spec(\mathbb{C})) \cong h^*[\tau].$$

Because the spectral sequence collapses, since $h^{*,*}(pt) \cong \mathbb{Z}[\tau], \ deg(\tau) = (0,1)$.

4. ABP-theory and the differntials for AHss.

Hu-Kriz [Hu-Kr2] has also shown that

$$(4.1) \quad MGL^{*,*}(MGL) \cong MGL^{*,*}(BGL) \cong MGL^{*,*}(pt)[c_1, ...].$$

This means that for AHss $E(X)_2^{*,*,*} = H^{*,*}(X, MU^*) \Longrightarrow MGL^{*,*}(X)$, we have the isomorphism of spectral sequences

$$(4.2) \quad E(MGL)_r^{*,*,*} \cong E(pt)_r^{*,*,*} \otimes H^*(MU).$$

Moreover the Steenrod algebra of MGL-theory is generated as an $MGL^{*,*}(pt)$-module by the Landweber-Novikov operation S_α correspondiong $c^\alpha = c_1^{\alpha_1} c_2^{\alpha_2} ...$ for $\alpha = (\alpha_1, \alpha_2, ...)$.

Hu-Kriz [Hu-Kr2] and Vezzosi [Ve2] constructerd ABP-theory by using the universal p-typical formal laws. However we will show here that the ABP-theory is also constructed by using the Lanweber-Novikov operations. These operations

satisfy the Cartan formula $S_\alpha(xy) = \Sigma_{\alpha=\beta+\gamma} S_\beta(x)S_\gamma(y)$ for $x, y \in MGL^{*,*}(X)$. Define an operation

$$(4.3) \quad \Delta_{x_i} = \Sigma_{q\geq 1}(x_i/S_{\Delta_i}(x_i))^{q-1} S_{q\Delta_i}.$$

Note that $\Delta_{x_i}(x_i) = 1$ if $i \neq p^j-1$. Then we can easily prove that $\pi_i = 1-x_i\Delta_{x_i}$ is a multiplicative projection such that $\pi_i(x_j) = (1-\delta_{ij})x_j$. Essentially composing (for details, see p587 in [No]) π_i for all $i \neq p^j - 1$, we get the multiplicative projection $\Phi : MGL^{*,*}(-)_{(p)} \to MGL^{*,*}(-)_{(p)}$ such that

$$(4.4) \quad \Phi(x_i) = \begin{cases} x_i & (if\ i = p^j - 1\ for\ some\ j) \\ 0 & (otherwise) \end{cases}$$

Let us write

$$\Phi MGL^{*,*}(X) = Im(\Phi(MGL^{*,*}(X)_{(p)}) \subset MGL^{*,*}(X)_{(p)}.$$

Since $H^*(BP;\mathbb{Z}) \cong H^*(\Phi MU_p;\mathbb{Z})$, we know $H^{*,*}(ABP;\mathbb{Z}) \cong H^{*,*}(\Phi MGL;\mathbb{Z})$ by using $w(x) = 0$ for $x \in \tilde{RP} \otimes \mathbb{Z}[m_i]$. Thus we have the isomorphism of spectra

$$(4.5) \quad ABP \cong \Phi MGL$$

by the Whitehead theorem in $\mathbb{A}^1$-stable homotopy category.

Lemma 4.1. *The theory $ABP^{*,*}(-)$ is a multiplicative theory such that*

$$ABP^{*,*}(X) \cong MGL^{*,*}(X)_{(p)} \otimes_{MU^*_{(p)}} BP^*.$$

Proof. Since $\pi_{x_i}(a) = (1-x_i\Delta_{x_i})a = a\ mod(x_i)$, we see $\Phi(a) = a\ mod(x_i|i \neq p^j-1)$ for all $a \in MGL^{*,*}(X)_{(p)}$. $\square$

Since $\tilde{RP} \cong \{r_\alpha|\alpha = (\alpha_1, \alpha_2, ...), \alpha_i \geq 0, \}$, the Seenrod algebra of ABP-theory is generated as an $ABP^{*,*}(pt)$-module by the Quillen operation r_α for $\alpha = (a_1, ...)$,i.e.

Proposition 4.2. *There is the isomorphism*

$$ABP^{*,*}(ABP) \cong ABP^{*,*}(pt) \otimes \tilde{RP}.$$

Proof. This is proved by using AHss

$$E(ABP)_2^{*,*,*} = H^{*,*}(ABP; BP^*) \Longrightarrow ABP^{*,*}(ABP).$$

Since $w(\tilde{RP}) = 0$, all elements in $\tilde{RP}$ are permant cycles. Hence we get the isomorphisms of spectral sequences $E(ABP)_r^{*,*,*} \cong E(pt.)_r^{*,*,*} \otimes \tilde{RP}$. $\square$

Next consider the differentials for $AP(n)^{*,*}(-)$ theory. It is known that the first nonzero differntial of AHss converging to $P(n)^*(-)$ is $d_{2p^n-1}(x) = v_n \otimes Q_n(x)$. By the naturality, for $AP(n)$-theory, the first nonzero differential

$$v_n^{-1} \otimes d_{2p^n-1} : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2p^n-1,*+p^n-1}(X;\mathbb{Z}/p)$$

is cohomology operation. We still know $t_{\mathbb{C}}(v_n^{-1} \otimes d_{2p^n-1}(x)) = Q_n(x)$. ¿From (2.18), the motivic Steenrod algebra $A_p^{*,*}$ is multiplicatively generated by $H^{*,*}(pt;\mathbb{Z}/p)$,P^i and Q_n. Hence we have

$$(4.6) \quad d_{2p^n+1}(x) = v_n \otimes (Q_n + a_{IJ}P^IQ_J)(x)$$

where $P^I \in RP, Q_J \in \Lambda(Q_0, ..., Q_{n-1}), |J| \geq 2,\ a_{IJ} \in H^{plus,*}(pt.\mathbb{Z}/p)$.

Lemma 4.3. *For all IJ, elements $a_{IJ} = 0$ in (4.6) if $n = 1$ or $k = \mathbb{C}$.*

Proof. When $n = 1$, the first degree of $v_1^{-1}d_{2p-1}$ is $2p-1$ and the weight is -1, we get

$$v_1^{-1}d_{2p-1} = Q_1 + aQ_0 \quad a \in H^{*,*}(pt;\mathbb{Z}/p).$$

But $a \in H^{2p-2,p-1}(pt;\mathbb{Z}/p) = 0$. When $k = \mathbb{C}$, thinking the realization map $t_{\mathbb{C}}$, we have the second assersion. $\square$

Corollary 4.4. *In the spectral sequence for ABP, we have*

$$d_{2p^n-1}(x) = v_n \otimes (Q_n + \sum a_{IJ}P^IQ_J) \qquad mod(M_n)$$

where M_n is a subalgebra of $E_{2p^n-1}^{*,*,minus}$ which is the subquotient module of $(p,...,v_{n-1})E_2^{*,*,0}$.

Proof. Consider the map of spectral sequences

$$E(ABP)_{2p^n-1}^{*,*,*} \to E(AP(n))_{2p^n-1}^{*,*,*}.$$

Here $E(AP(n))_{2p^n-1}^{*,*,*} \cong E(AP(n))_2^{*,*,*}$ since the first nonzero differential for $AP(n)$-theory is d_{2p^n-1}. Hence

$$E(ABP)_{2p^n-1}^{*,*,*}/M \to E(AP(n))_{2p^n-1}^{*,*,*}$$

is injective and we have the corollary. $\square$

5. $(p, v_1, ..., v_n)$-torsion spaces

In this section, we consider $(p, v_1, ..., v_n)$-torsion spaces and their applications according to V.Voevodsky. Recall that $BP\langle n\rangle^*(X)$ be the cohomology theory with the coefficient $BP\langle n\rangle^* = \mathbb{Z}_{(p)}[v_1, ..., v_n]$ so that $BP\langle -1\rangle^*(X) = H^*(X;\mathbb{Z}/p)$ and $BP\langle\infty\rangle^*(X) = BP^*(X)$.

Lemma 5.1. *If $x = Q_n...Q_1Q_0x'$ in $H^{*,*}(X;\mathbb{Z}/p)$, then $x \in E(ABP)_{2p^n}^{*,*,0}$ and x is $(2, v_1, .., v_n)$-torsion in $E(ABP)_{2p^n}^{*,*,*}(X)$.*

Proof. There is the cofiber map of spectra

$$T^{p^k-1} \wedge ABP\langle n\rangle \xrightarrow{v_k} ABP\langle k\rangle \xrightarrow{\rho_k} ABP\langle k-1\rangle \xrightarrow{\delta_k}$$

Consider the (Sullivan-Bockstein) exact sequece, namely, the long exact sequence induced from the above cofiber map

$$ABP\langle k\rangle^{*+2p^k-2,*+p^k-1}(X) \xrightarrow{v_k} ABP\langle k\rangle^{*,*}(X) \xrightarrow{\rho_k}$$
$$ABP\langle k-1\rangle^{*,*}(X) \xrightarrow{\delta_k} ABP\langle k\rangle^{*+2p^k-1,*+p^k-1}(X).$$

The induced map

$$Im(ABP\langle n-1\rangle^{*,*}(X) \to H^{*,*}(X;\mathbb{Z}/p)) \to H^{*,*}(X:\mathbb{Z}/p)$$

defined by $\rho_0...\rho_{n-1}(x) \mapsto \rho_0...\rho_n\delta_n(x)$ represents the operation $Q_n + \sum a_{IJ}P^IQ_J$ with $a_{IJ} \in H^{plus,*}(X;\mathbb{Z}/p)$ and $J \geq 2$ from the topological case [Sh-Ya].

By the Baas-Sullivan exact sequence, we can see that $x'' = \delta_n...\delta_0(x') \in ABP\langle n+1\rangle^{*,*}(X)$ is $(2, v_1, ..., v_n)$-torsion since the map δ_i is a map of AMU-module spectra. In particular, $x = Q_n...Q_0(x') = \rho_0...\rho_n(x'')$ is a permanent cycle in the spectral sequence

$$E(ABP\langle n\rangle)_2 = H^{*,*}(X; BP\langle n\rangle^*) \Longrightarrow ABP\langle n\rangle^{*,*}(X),$$

and $d_{2p^n-1}(y) = v_n \otimes x$ for $y = Q_{n-1}...Q_0(x')$. Compare with the spectral sequence

$$E(ABP)_2^{*,*,*} \cong H^{*,*}(X; BP^*) \Longrightarrow ABP^{*,*}(X).$$

Since $BP^* \cong BP\langle n\rangle^*$ for $* > -2p^{n+1}+1$, we see $E(ABP)_2^{*,*',*''} \cong E(ABP\langle n\rangle)_2^{*,*',*''}$ for $*'' > -2p^{n+1}+1$. Hence the map $i^{*,*',*''}: E(ABP)_r^{*,*',*''} \to E(BP\langle n\rangle)_r^{*,*',*''}$ is injective for $r < 2p^{n+1}-1$ and $*'' > -2p^{n+1}+1$. In particular we see that x is also $(2,...v_n)$-torsion in $E(ABP)_{2^{n+1}}^{*,*,*}$. □

Corollary 5.2. *If $H^{*,*}(X;\mathbb{Z}/p) \supset \oplus\Lambda(Q_n,...,Q_1)G_n$ with $w(G_n)=n$, then*

$$grAP(1)^{2*,*}(X) \supset \oplus P(n+1)^*Q_1...Q_nG_n.$$

If $H^{*,*}(X;\mathbb{Z}/p)$ is $\Lambda(Q_0,...,Q_n)$-free and $H^{*,*}(X)$ is just p-torsion, we have more strong results. Moreover the converse also holds.

Lemma 5.3. *If $ABP\langle k\rangle^{*,*}(X)$ is $(p,v_1,...v_k)$-torsion for all $k \le n$, then $H^{*,*}(X;\mathbb{Z}/p)$ is a free $\Lambda(Q_0,...,Q_n)$-module. If all nonzero elements in $H^{*,*}(X;\mathbb{Z})$ are just p-torsion then the converse is also holds.*

Proof. Consider the Baas-Sullivan exact sequence in the proof of the preceding lemma. Here $v_k=0$ and we have the short exact sequence. Letting $\delta_k\rho_k = Q_k'$, we can write

$$ABP\langle k-1\rangle^{*,*}(X) \cong \Lambda(Q_k') \otimes ABP\langle k\rangle^{*,*}(X).$$

Since $ABP\langle -1\rangle^{*,*}(X) \cong H^{*,*}(X;\mathbb{Z}/p)$, we get the result by induction

$$H^{*,*}(X;\mathbb{Z}/p) \cong \Lambda(Q_0,...,Q_n) \otimes ABP\langle n\rangle^{*,*}(X).$$

Conversely let $ABP\langle k-1\rangle^{*,*}(X) \cong \Lambda(Q_k') \otimes G$. Recall again $\rho_k\delta_k G = Q_k'G$ and $\delta_k|G$ is injective. From the above exact sequence

$$ABP\langle k\rangle^{*,*}(X)/(v_k) = Image(\delta_k) = Q_k'G.$$

Hence $ABP\langle k\rangle^{*,*}(X)$ is generated as a $\mathbb{Z}/p[v_k]$-module by δ_kG. Thus $ABP\langle k\rangle^{*,*}(X)$ is v_k-torsion and it is isomorphic to $Q_k'G$. By induction, we can prove if $H^{*,*}(X;\mathbb{Z}/p) \cong \Lambda(Q_0,...Q_n) \otimes G$ then

$$ABP\langle n\rangle^{*,*}(X) \cong Q_n...Q_0G\ ;\ it\ is\ (p,v_1,...,v_n)-torsion.$$

□

Let the $\check{C}ech$ complex $\check{C}(X)$ be the simplicial scheme such that $\check{C}(X)_n = X^{n+1}$ and the faces and degeneracy maps are given by partial projections and diagonals respectively ([Vo1,2]). One of the important properties of $\check{C}(X)$ is the following.

Lemma 5.4. *([Vo1]) Let X,Y be smooth schemes such that $Hom(Y,X) \ne \emptyset$. Then the projection $\check{C}(X)\times Y \to Y$ is a simply weak equivalence.*

In the stable $\mathbb{A}^1$ homotopy category, define $\tilde{C}(X)$ by the following cofiber sequence

$$(5.1) \quad \tilde{C}(X) \to \check{C}(X) \to Spec(k).$$

Lemma 5.5. *(([Vo1]) Let $p: Y \to Spec(k)$ be the projection and $t_{\mathbb{C}}(p_*([Y])) = y$ in BP^*. Let $Ah = ABP(S_n)$ for some regular sequence S_n in BP^*. If $Hom(Y,X) \ne \emptyset$, then $Ah^{*,*}(\tilde{C}(X))$ is y-torsion.*

Proof. For $z \in Ah^{*,*}(X \wedge Y)$ and $[Y] \in ABP_*(Y)$, define the map p_{Y*} by

$$p_{Y*}: X \wedge T^{|y|} \xrightarrow{1\wedge[Y]} X \times Y \wedge ABP \xrightarrow{z\wedge 1} Ah \wedge ABP \xrightarrow{\mu} Ah.$$

Then the composion of maps

$$p_{Y*}p^*: Ah^{*,*}(X) \to Ah^{*,*}(X\times Y) \to Ah^{*+|y|,*+1/2|y|}(X)$$

induces $p_{Y*}p^*(x) = yx$. But $Ah^*(\tilde{C}(X) \times Y) \cong 0$ since $Ah^{*,*}(\check{C}(X) \times Y) \cong Ah^{*,*}(Y)$. $\square$

Corollary 5.6. *Suppose that there are maps $V_0 \to V_1 \to ... \to V_n$ such that $t_{\mathbb{C}}(p_*[V_i]) = v_i$ for all $i \leq n$. Then $H^{*,*}(\tilde{C}(V_n); \mathbb{Z}/p)$ is a free $\Lambda(Q_0, ..., Q_n)$-module.*

V.Voevodsky applied above corollary by the following situation. Suppose that $p = 2$. For $a = \{a_0, ..., a_n\}, a_i \in k^*$ and $0 \neq a \in K^M_{n+1}(k)$, let Q_a be the norm quadric, namely the projective quadric of dimension $2^n - 1$ defined by the form

$$(5.2) \quad q_a =<< a_0, ..., a_{n-1} >> - < a_n > .$$

Here $<< a_0, ..., a_{n-1} >>=< 1, -a_0 > \otimes ... \otimes < 1, -a_{n-1} >$ is the $n+1$-fold Pfister form. Let us write $\check{C}(Q_a) = \chi_a$ and $\tilde{C}(Q_a) = \tilde{\chi}_a$. Note that $H^{*,*'}(\chi_a; \mathbb{Z}/2) \cong H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/2)$ when $* > *'$. (Moreover $H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/2) \cong 0$ for $* \leq *'$ by the solution of this Milnor conjecture.)

It is known ([Vo1], Rost[Ro]) that

$$(5.3) \quad t_{\mathbb{C}}(p_*([Q_a]) = v_n, \qquad H^{2^{n+1}-1, 2^n}(\chi_a; \mathbb{Z}/2) = 0.$$

Since $H^{n+2,n+1}(\chi_a; \mathbb{Z}/2) \cap Im(Q_i) = 0$ by inductive assumption for dimensional reason, we know that the map

$$Q_1 ... Q_{n-1} : H^{n+2,n+1}(\tilde{\chi}_a; \mathbb{Z}/2) \to H^{2^{n+1}-1,2^n}(\tilde{\chi}_a; \mathbb{Z}/2)$$

is injective by the $\Lambda(Q_0, ..., Q_n)$-freeness. Note also $H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/2) \cong H^{*,*}(\chi_a; \mathbb{Z}/2)$ for these degree $(*, *')$. Hence from (5.3), we see $H^{n+2,n+1}(\chi_a; \mathbb{Z}_{(2)}) = 0$ because we already knew that all elements in $H^{*,*}(\chi_a)_{(2)}$ are just 2-torsion. This is one of the key lemma for the proof of Milnor conjecture by V.Voevodsky.

In the category of Chow motives, Rost [Ro] constructed a direct summand M_a of Q_a such that if q_a has a K-rational point for an extension K of k, then

$$(5.4) \quad (M_a)_K = T_K^0 \oplus T_K^{\otimes(2^n-1)}$$

where $T_K = T \otimes_k Spec(K)$ is the Tate motive for K. If q_a has no rational point, then the Chow group is

$$(5.5) \qquad CH^m(M_a) \cong \begin{cases} \mathbb{Z} & if\ m = 0 \\ \mathbb{Z}/2 & if\ m = 2^n - 2^k\ for\ 1 \leq k \leq n-1 \\ 0 & otherwise. \end{cases}$$

V.Vevodsky proved that in the category DM_-^{eff} of geometric motives ([Vo1] for details), there exists the distiguished triangle

$$M(\chi_a)(2^n - 1)[2^{n+1} - 2] \to M_a \to M(\chi_a) \to M(\chi_a)(2^n - 1)[2^{n+1} - 1]$$

where $M(\chi_a)$ is the motive of χ_a and $(*')[*]$ the operator of changing degree, namely, there is the long exact sequence

$$(5.6) \quad \leftarrow H^{*-2^{n+1}+2,*-2^n+1}(M(\chi_a); \mathbb{Z}/2) \leftarrow H^{*,*}(M_a; \mathbb{Z}/2)$$
$$\leftarrow H^{*,*}(M(\chi_a); \mathbb{Z}/2) \overset{\delta_a}{\leftarrow} H^{*-2^{n+1}+1,*-2^n+1}(M(\chi_a);\ bZ/2) \leftarrow .$$

Denote by $k(Q_a)$ the function field of Q_a and by $(Q_a)_0$ the set of closed points of Q_a. The main theorem of the paper [Or,Vi,Vo] by Orlov,Vishik and Voevodsky is the following.

Theorem 5.7. *([Or, Vi, Vo]) F any $a = (a_0, ..., a_n)$ in k^*, the following sequence is exact*

$$\amalg_{x \in (Q_a)_0} K_*^M(k(x))/2 \xrightarrow{Tr_{k(x)/k}} K_*^M(k)/2 \xrightarrow{a} K_{*+n+1}^M(k)/2 \to K_{*+n+1}^M(k(Q_a))/2.$$

The outline of the proof by Orlov, Vishik and Voevodsky is as follows. We first see the two exact sequences

$$(5.7) \quad 0 \to H^{*+1,*}(\chi_a;\mathbb{Z}/2) \to K_{*+1}^M(k)/2 \to K_{*+1}^M(k(Q_a))/2.$$

$$(5.8) \ \amalg_{x \in (Q_a)_0} K_*^M(k(x))/2 \xrightarrow{Tr_{k(x)/k}} K_*^M(k)/2 \xrightarrow{\delta_a} H^{*+2^{n+1}-1,*+2^n-1}(\chi;\mathbb{Z}/2) \to 0.$$

Next step is to prove that the map

$$(5.9) \quad K_*^M(k)/2 \xrightarrow{\delta_a} H^{*+2^{n+1}-1,*+2^n-1}(\chi_a;\mathbb{Z}/2)$$
$$\xrightarrow{(Q_{n-1}...Q_0)^{-1}} H^{*+n+1,*+n}(\chi_a;\mathbb{Z}/2) \to K_{*+n+1}^M(k)/2$$

is the multiplication with a. To see the existence of $(Q_{n-1}...Q_0)^{-1}$, Corollary 5.6 is used. Here we note the case $* = 0$.

$$\mathbb{Z}/2 \cong K_0(k)/2 \to H^{2^{n+1}-1,2^n-1}(\chi_a;\mathbb{Z}/2) \xrightarrow{(Q_{n-1}...Q_0)^{-1}} H^{n+1,n}(\chi_a;\mathbb{Z}/2) \to K_{n+1}^M(k)/2.$$

The image of the maps in the righthand side term is generated by a from (5.7), since Q_a is a splitting variety of a. Above maps are all $K_*^M(k)$-module maps and we get the results.

Theorem 5.8. *Let $0 \neq a = (a_0, ..., a_n) \in K_{n+1}^M(k)$. Then there is a $\Lambda(Q_0, ..., Q_n)$-modules isomorphism*

$$H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2) \cong K_*^M(k)/(Ker(a)) \otimes \Lambda(Q_0, ..., Q_n) \otimes \mathbb{Z}/2[\delta_a^2]\{a\tau^{-1}\}$$

where $deg(\delta_a^2) = (2^{n+2}-2, 2^{n+1}-2)$, $deg(a\tau^{-1}) = (n+1, n)$.

Proof. Recall the difference $d(x) = first.deg(x) - second.deg(x)$. Hence if $0 \neq x \in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$, then $d(x) > 0$. Since $dim(Q_a) = 2^n - 1$, we also know if $0 \neq x \in H^{*,*}(M_a;\mathbb{Z}/2)$, then $d(x) \leq 2^n - 1$. ¿From the exact sequence(5.6), if $0 \neq x \in H^{*,*}(\chi_a;\mathbb{Z}/2)$ and $d(x) \geq 2^n$, then the map δ_a is an epimorphism, namely, there is an element $\delta_a(x)^{-1} \in H^{*,*}(\chi_a;\mathbb{Z}/2)$.

Let us write $\Lambda = \Lambda(Q_0, ..., Q_n)$ simply. We prove the theorem by induction on $d(t)$ for a Λ-module generator t in $H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$. From (5.9) we already know that

$$K_*^M(k)/2 \xrightarrow{\delta_a} H^{*+2^{n+1}-1,*+2^n-1}(\chi;\mathbb{Z}/2) \xrightarrow{(Q_{n-1}...Q_0)^{-1}} H^{*+n+1,*+n}(\chi_a;\mathbb{Z}/2)$$

is an epimorphism, indeeed, the map δ_a is epic from (5.6) and the map $Q_{n-1}...Q_0$ is isomorphic since $H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ is Λ-free. The composition of the above map with $H^{*+n+1,*+n}(\tilde{\chi}_a;\mathbb{Z}/2) \to K_{*+n+1}^M(k)/2$ is multiplying a from (5.9). Since the last map is monic from (5.7), we see that

$$H^{*+n+1,*+n}(\tilde{\chi}_a;\mathbb{Z}/2) \cong K_*^M(k)/(Ker(a))\{a\} \subset K_{*+n+1}^M(k).$$

Hence we get the case $d(t) = 1$.

Suppose that the isomorphism in the theorem holds for degree $d(x) < d$. Let $t \in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ be a smallest weight element such that it is a Λ-module generator with $d(t) = d$. Then we see

$$d(Q_0...Q_{n-1}t) = 2^0 + ... + 2^{n-1} + d = 2^n - 1 + d \geq 2^n$$

Hence there is an element $y=\delta_a^{-1}(Q_{n-1}...Q_0t)$. Since $d(\delta_a)=2^n$, we have

$$d(y)=2^n-1+d-2^n=d-1>0.$$

This means $y\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$.

Then on $H^{deg(y)}(\tilde{\chi}_a;\mathbb{Z}/2)$, δ_a is epic but Q_n is not epic since $Q_n(\delta_a y)=Q_0...Q_nt\neq 0$. Hence there is a nonzero element $y'\in H^{deg(y)}(\tilde{\chi}_a;\mathbb{Z}/2)$ such that $Q_ny'=0$. By the Λ-freeness, there exists $z\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ such that $Q_n(z)=y'$.

Next consider the difference of t

$$\begin{aligned}d(t)&=d(Q_0...Q_{n-1}t)-d(Q_0...Q_{n-1})=d(\delta_aQ_nz)-d(Q_0...Q_{n-1})\\&=d(z)+2^{n+1}-(2^n-1)=d(y')+2^n+1>2^n+1\end{aligned}$$

since $0\neq y'\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ and $d(z)>0$. Thus we can also define $\delta_a^{-1}(t)\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$.

Moreover $Q_i(t)\neq 0$ implies $0\neq \delta_a^{-1}(Q_i(t))$ for all $0\leq i\leq n-1$. Since $Q_n(t)\neq 0$, we can also prove that there is the element $0\neq u\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ such that $deg(Q_n(u))=deg(\delta_a^{-1}(t))$.

By induction, each nonzeo element $x\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ with $d(x)<d$ is represented as $x=bQ_{i_1}...Q_{i_s}\delta_a^{2\ell}\tau^{-1}a$ for $b\in K_M^*(k)$. Hence its difference degree is

$$d(x)=2^{i_1}+...+2^{i_s}+2^{n+1}\ell+1 \qquad .$$

This implies that for each $0\leq i\leq n-1$, if there is nonzero elements u', $t_i\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ such that $d(t_i)=d(Q_nu')+2^i$, then u' is a Λ-module generator. Thus we know u is a Λ-module generator.

Suppose that u is not a $K_*^M(k)/2$-module generator. Then there is a nonzero element $u'\in H^{*,*}(\tilde{\chi}_a;\mathbb{Z}/2)$ with $w(u')<w(u)$. Then δ_a^2u' is a Λ-module generator of $w(\delta_a^2u')<w(t)$. This contradicts to the assumption of t. Hence u is also a $K_*^M(k)/2$-module generator. By inductive assumption, we know $H^{deg(u)}(\tilde{\chi}_a;\mathbb{Z}/2)\cong\mathbb{Z}/2$. Thus we know $\delta_a^2(u)=t$. $\square$

Corollary 5.9. *There is a $K_*^M(k)$-module isomorphism for $first.deg>second.deg$ and $0<second.deg<2^n-1=dim(Q_a)$,*

$$H^{*,*}(M_a;\mathbb{Z}/2)\cong K_*^M(k)/(2,Ker(a))\otimes\Lambda(Q_0,...Q_{n-1})\{a\tau^{-1}\}.$$

Remark The formula (5.5) is the immedite consequence of the above theorem, while (5.3) has been used to prove the theorem. Moreover Rost proved in [Ro] (Theorem 6 in [Ro])

$$H^{2s-1,s}(M_a,\mathbb{Z})\cong\begin{cases}(1)\ \ \mathbb{Z}/2\ or\ 0 \quad if\ s=2^n-2^k-2^\ell+1\\(2)\ \ a\ subgroup\ of\ k^* \quad if\ s=2^n-1\\(3)\ \ a\ quotient\ of\ CH^{s-1}(M_a)\otimes k^* \quad for\ other\ s.\end{cases}$$

Hence Theorem 5.5 extends the cases (1) and (3), however (2) is used to prove (5.3).

6. Real field case

In this section, we consider the case $p=2$ and $k=\mathbb{R}:$ the field of real numbers. We first recall the Milnor K-theory

$$K_*^M(\mathbb{R})/2\cong\mathbb{Z}/2[\rho]\quad with\quad \rho=-1\in\mathbb{R}^*/(\mathbb{R}^*)^2=K_1^M(\mathbb{R})/2.$$

Hence the mod 2 motivic cohomology is

$$H^{*,*} = H^{*,*}(pt;\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau] \quad with \;\; deg(\rho)=(1,1),\; deg(\tau)=(0,1).$$

We want to study Q_n-action on $H^{*,*}$. The problem is that Q_i is not a derivation.

Lemma 6.1. *(Voevodsky [Vo3] Prop.13.4) Let ψ be the coproduct for the mod 2 motivic Steenrod algebra $A_2^{*,*}$. Then*

$$\psi(Q_n) = Q_n \otimes 1 + 1 \otimes Q_n + \rho Q_{n-1} \otimes Q_{n-1} \quad mod(A_n)$$

$$where \quad A_n = \{\sum \rho a Q_i \otimes b Q_j | a,b \in A_2^{*,*}, 0 \le i,j \le n-1 \; but (i,j) \ne (n-1,n-1)\}.$$

Proof. Let $A_{2,*,*}$ be the dual of $A_2^{*,*}$. Then by Voevodsky (Theorem 12.6), there is the additive isomorphism $A_{2,*,*} \cong H^{*,*} \otimes \mathbb{Z}/2[\xi_i] \otimes \Lambda(\tau_i)$ with the multiplication

$$\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1}.$$

Here τ_i is the dual of Q_i. The lemma follows this equation. □

Lemma 6.2. *Let $t_k = \tau^{2^k}$ and let $grH^{*,*} = \mathbb{Z}/2[\rho] \otimes \Lambda(t_0, t_1, ...)$. Then Q_n acts on $H^{*,*}$ as a derivation on $grH^{*,*}$ with $Q_n(\rho)=0$, $Q_n(t_n) = \rho^{2^{n+1}-1}$, $Q_n(t_j) = 0$ for $n \ne j$, namely, $Q_n(\Pi t_i^{e_i}\rho^k) = (\Pi_{n\ne i} t_i^{e_i})(e_n\rho^{2^{n+1}-1})\rho^k$.*

Proof. By dimensional reason, $Q_n(\rho^k)=0$ and $Q_n(\Pi_{i<n} t_i^{e_i}) = 0$. By the definition of τ and ρ, we see $Q_0(\tau)=\rho$. Since Q_0 is derivation, we have $Q_0(\tau^{2m}) = 0$ for $m \ge 1$. By induction we assume that

$$Q_j(t_m) = 0 \; for \; j \ne m < n, \quad and \;\; Q_m(t_m) = \rho^{2^{m+1}-1} \; for \; m < n.$$

In $H^{*,*}$, we get from the preceeding lemma

$$Q_n(t_n) = Q_n(t_{n-1}^2) = \rho Q_{n-1}(t_{n-1})Q_{n-1}(t_{n-1}) + \sum \rho a_i Q_i(t_{n-1}) b_i Q_j(t_{n-1}).$$

By inductive assumption, we see $Q_i(t_{n-1}) = 0$ or $Q_j(t_{n-1}) = 0$. Hence we see

$$Q_n(t_n) = \rho Q_{n-1}(t_{n-1})Q_{n-1}(t_{n-1}) = \rho^{2^{n+1}-1}$$

by also inductive assumption. By the similar arguments we also see $Q_n(t_j) = 0$ for $j > n$. ¿From these we can get the last statement in this lemma by the similar arguments. □

Lemma 6.3. *The reduced powers Sq^{2^n} acts on $H^{*,*}$ as a derivation on $grH^{*,*}$ with $Sq^{2^n}(\rho) = 0$, $Sq^{2^n}(t_n) = t_{n-1}\rho^{2^n}$, $Sq^{2^n}(t_j) = 0$ for $n \ne j$.*

Proof. It is known from ([Vo3],Lemma 9.8),

$$Sq^{2i}(x) = 0 \qquad if \; i > d(x) \; and \; i \ge second.deg(x).$$

Hence $Sq^{2^n}(t_j) = 0$ for $n > j$.

Veovodsky ([Vo3] Lemma 9.6) also proves the Cartan formula for motivic cohomology operations

$$Sq^{2i}(xy) = \sum_r Sq^{2r}(x)Sq^{2i-2r}(y) + \tau Sq^1Sq^{2s}(x)Sq^1Sq^{2i-2-2s}(y).$$

When $n < j$, we have

$$Sq^{2^{n+1}}(t_{j+1}) = (Sq^{2^n}(t_j))^2 + \tau(Sq^1Sq^{2^n-2}(t_j))^2$$

since Sq^m is a sum of products of Sq^{2^i}, we know $Sq^n(t_j) = 0$ by induction.

When $n = j$, we also have the result by induction and

$$Sq^2(\tau^2) = 2Sq^2(\tau)\tau + \tau Sq^1(\tau)Sq^1(\tau) = \tau\rho^2.$$

□

Theorem 6.4. *We have the isomorphism with $deg(t_0) = (0,1)$, $deg(t_{n+1}) = (0, 2^{n+1})$*

$$grAk(n)^*(Spec(\mathbb{R})) \cong k(n)^* \otimes \mathbb{Z}/2[t_0, t_{n+1}, \rho]/(t_0^{2^n}, v_n\rho^{2^{n+1}-1})$$

Proof. ¿From (4.6) the first nonzero differential is

$$d_{2^{n+1}-1}(x) = v_n \otimes (Q_n + \sum a_{IJ}P^I Q_J)(x).$$

By the dimensional reason, the operation P^I, Q_J are sums of products of Sq^{2^i}, $Q_{i'}$ for $i, i' < n$ respectively.

Let $x = t't''$ with $t' \in \Lambda(t_0, ..., t_{n-1})$ and $t'' \in \Lambda(t_n, t_{n+1}, ...)$. By the above lemmas and the Cartan formula, we have $P^I Q_J(t't'') = (P^I Q_J t')t''$. The difference is

$$d(P^I Q_J t') = d(Q_n) - d(a_{IJ}) + d(t') > 2^n - d(a_{IJ}) - 2^n > 0$$

since $d(a_{IJ}) < 0$ and $d(t') > -2^n$. But if $0 \neq y \in H^{*,*}$ then $d(y) \leq 0$. Hence $P^I Q_J t' = 0$ and so $P^I Q_J x = 0$. Thus we get

$$d_{2^{n+1}-1}(x) = v_n \otimes Q_n(x).$$

The fact that $Q_n t_n = \rho^{2^{n+1}-1}$ implies that $E_{2^{n+1}}^{*,*}$ is isomorphic to the righthand module in the theorem. Since $E_r^{*,*',minus} \cong 0$ if $* \geq 2^{n+1} - 1$, we see that $d_r = 0$ for all $r \geq 2^{n+1}$. Thus $E_{2^{n+1}}^{*,*}$ is isomorphic to the infinitive term of AHss. □

Remark. Let $K_*(k; \mathbb{Z}/2)$ be the algebraic mod 2 K-group of the field k. Then it is known

$$K_*(k; \mathbb{Z}/2) \cong K(1)_*(Spec(k))/(v_1 = 1).$$

Hence we have

$$grK_*(\mathbb{R}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho, t_0, t_2]/(t_0^2, \rho^3) \cong \mathbb{Z}/2[t_2]\{1, \rho, (\rho^2, t_0), t_0\rho, t_0\rho^2, 0, 0, 0\}.$$

Here the degree n of the algebraic K-group $K_n(-; \mathbb{Z}/2)$ is given by the weight of the motivic K-theory, namely, $w(t_2) = 8, w(\rho) = 1, w(t_0) = 2$. This is the famous result of Suslin [Su1] for the mod 2 algebraic K-theory of $\mathbb{R}$.

P.Hu and I.Kriz computed the coefficient of the Real cobordism theory and Real Morava K-theories. By conjugation, $\mathbb{Z}/2$ acts on $BU(n)$ and on the Thom space $Th(BU(n))$ of the universal complex n-bundle. Then we can define the suspension map

$$S^{\mathbb{C}}Th(BU(n)) \to Th(BU(n+1)).$$

Here $\mathbb{Z}/2$-equivariantly, we identify $\mathbb{C} = (1,1) + (1,0)$ where $(1,0)$ is the trivial *one*-dimensional representation of $\mathbb{Z}/2$, and $(1,1)$ is the sign representation. Then we can define (for details see Greenlees [Gre] or [Hu-Kr1]) a spectrum $M\mathbb{R}$ and the generalized cohomology theory $M\mathbb{R}^{*,*}(X)$ for $\mathbb{Z}/2$-equivariant spaces X. When $k \subset \mathbb{R}$, we can define a Real realization map $t_{\mathbb{R}}$ (see also [Vo1]),

$$t_{\mathbb{R}} : MGL^{*,*}(X) \to M\mathbb{R}^{*,*}(t_{\mathbb{R}}(X))$$

where $t_{\mathbb{R}}(X)$ is the space $X(\mathbb{C})$ with Galois action of $\mathbb{Z}/2 \cong Gal(\mathbb{C}/\mathbb{R})$.

Similary, the Real BP-theory $BP\mathbb{R}^{*,*}(-)$ and the Real Morava K-theory $K(n)\mathbb{R}^{*,*}(-)$ are constructed.

By Greenlees and Hu-Kriz, the coefficcient of the Real ordinaly cohomology $H\mathbb{R}$ is

$$H\mathbb{R}^{*,*}(pt.;\mathbb{Z}/2)\cong \mathbb{Z}/2[\rho,\tau,\tau^{-1}]\cong H^{*,*}[\tau^{-1}].$$

Moeover from Theorem 6.40 in [Hu-Kr1] and Theorem 12.6 in [Vo3] we see that the dual of the Real Steenrod algebra is

$$A\mathbb{R}_{*,*}\cong A_{2,*,*}(\mathbb{R})[\tau^{-1},(\tau_0\rho+\tau)^{-1}]\cong A_{2,*,*}(\mathbb{R})[\tau^{-1}]^{\wedge}_{(\tau_0\rho/\tau)}.$$

Hu and Kriz ([Hu-Kr2]) have shown

$$grK(n)\mathbb{R}^{*,*}(pt.)\cong K(n)^*\otimes Z/2[t_0,t_{n+1},t_{n+1}^{-1},\rho]/(t_0^{2^n},\rho^{2^{n+1}-1})$$

Hence we get

Corollary 6.5. $AK(n)^{*,*}(Spec(\mathbb{R}))[t_{n+1}^{-1}]\cong K(n)\mathbb{R}^{*,*}(pt)$

Moreover Hu and Kriz have shown ([Hu-Kr1] Theorem 4.11)

$$BP\mathbb{R}^{*,*}(pt)\cong B(\rho).C(\tau)\subset B(\rho)\otimes\mathbb{Z}/2[t_2,t_2^{-1}]$$

$$where\quad B(\rho)=BP^*[\rho]/(2,v_n\rho^{2^{n+1}-1}|n\ge 1),\quad C(\tau)=\mathbb{Z}/2\{v_nt_2^{2^{n-1}\ell}|\ell\in\mathbb{Z}\}.$$

Theorem 6.6. *There are isomophisms*

$$gr(ABP/2)^{*,*}(Spec(\mathbb{R})\cong(BP\mathbb{R}^{*,*}(pt)\cap B(\rho)\otimes\mathbb{Z}/2[t_2])\otimes\Lambda(t_0)$$

$$\cong(B(\rho).\mathbb{Z}/2\{1,v_nt_2^{2^{n-1}\ell}|\ell\ge 1\})\otimes\Lambda(t_0)\subset(B(\rho)\otimes\mathbb{Z}/2[t_2])\otimes\Lambda(t_0).$$

Proof. Consider AHss

$$E(ABP/2)_2^{*,*,*}=H^{*,*}(pt;BP^*/2)\Longrightarrow(ABP/2)^{*,*}(pt).$$

At frst, we see $d_3(at_1)=a\rho^3$ for $a=\Pi_{i\ne 1}t_i^{e_i}\rho^k$. Hence we have

$$E_4\cong BP^*\otimes\Lambda(t_0,t_2,...)\otimes\mathbb{Z}/2[\rho]/(v_1\rho^3).$$

Here note that $v_1E_4^{*,*',*''}=0$ for $*\ge 3$. Let $A_1=0$.

By induction, we assume

$$(**)\quad E_{2^{n+1}}^{*,*,*}=A_n\oplus BP^*/2\otimes\Lambda(t_0,t_{n+1},...)\otimes\mathbb{Z}/2[\rho]/(v_1\rho^3,...,v_n\rho^{2^{n+1}-1})$$

where A_n is an $BP^*/2$-module with generators in $E_{2^{n+1}}^{*,*',minus}, 0\le *<2^{n+1}-1$. If $2^{n+1}\le r<2^{n+2}-1$, then $d_r=0$ because $d_r(x)=v_1^{a_1}...v_n^{a_n}c$ for $(a_1,...,a_n)\ne(0,...0)$ but by dimensional reason, this is zero. Hence the next nonzero differential is

$$d_{2^{n+2}-1}(x)=v_{n+1}Q_{n+1}(x)\ mod(2,...,v_n)\quad for\ x\in E_{2^{n+2}-1}^{*,*',0}$$

since $a_{IJ}P^IQ_J(x)=0$ for $x\in\Lambda(t_0,t_{n+1},..)$ from proceeding lemmas. In particular

$$d_{2^{n+2}-1}(at_{n+1})=v_{n+1}\rho^{2^{n+2}-1}a\quad for\ a\in\mathbb{Z}/2[\rho]\otimes\Lambda(t_0,t_{n+2},...).$$

Moreover $d_{2^{n+2}-1}(a)=0$ for $a\in A_n$ by the following reason. The map $i: ABP/2\to P(n+2)$ of spectra induces the map of the spectral sequences

$$i^*:E(ABP/2)_{2^{n+2}-1}^{*,*',*''}\to E(P(n+2))_{2^{n+2}-1}^{*,*',*''}=E(P(n+2))_2^{*,*',*''},$$

which is injective for $*>-2^{n+1}+1$. Since $i^*(a)=0$, we have $d_{2^{n+2}-1}(a)=0$. Thus we can prove the $(n+1)$-version of $(**)$.

Moreover we know that

$$A_{n+1}\cong A_n\oplus BP^*[\rho]/(v_1\rho^3,...,v_n\rho^{2^{n+1}-1}).B_{n+1}$$

$$where \quad B_{n+1} = \mathbb{Z}/2\{v_1t_{n+1}, ..., v_nt_{n+1}\} \otimes \Lambda(t_{n+2}, ...)$$

because $d_{2^{n+2}-1}(v_it_{n+1}) = 0$ for all $i < n+1$. Thus we know

$$A_\infty \cong \{v_nx | x \in \Lambda(t_{n+1}, ...), n \geq 1\}.BP^*[\rho]/(v_1\rho^3, ..., v_n\rho^{2^{n+1}-1}, ...).$$

Since $v_nt_{n+1} = v_nt_2^{2^{n-1}}$, we have the expression of the theorem.

□

7. Complex case

Throghout this section, we assume $k = \mathbb{C}$. At first we recall that $H^{*,*}(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau]$ and $Ah^{*,*}(pt) \cong h^* \otimes \mathbb{Z}/p[\tau]$ when h^* is $\mathbb{Z}/p$-module, namely, the mod p-theory.

Proposition 7.1. *(Thomason's type therorem) Let $h^*(-)$ be a mod p-theory. There is an isomorphism*

$$[\tau^{-1}]Ah^{*,*}(X) \cong h^*(X(\mathbb{C})) \otimes \mathbb{Z}/p[\tau, \tau^{-1}]$$

where bidegree of $x \in h^m(X(\mathbb{C}))$ is given by (m, m).

Proof. Consider AHss localized by $[\tau^{-1}]$

$$[\tau^{-1}]E_2^{*,*,*}(X) \cong [\tau^{-1}]H^{*,*}(X; \mathbb{Z}/p) \otimes h^* \Longrightarrow [\tau^{-1}]Ah^{*,*}(X).$$

Recall the Suslin's theorem [Su2] such that

$$H^{m,n}(X, \mathbb{Z}/p) \cong H^m_{et}(X, \mu_p^{\otimes n}) \quad for\ n \geq dim(X)$$

for an algebraic closed field k. Hence $H^{m,n}(X; \mathbb{Z}/p) \cong H^m(X(\mathbb{C}); \mathbb{Z}/p)$ for $n \geq dim(X)$. (If $B(n,p)$-condition holds, the isomorphism for $n \geq m$.) In any case we have $[\tau^{-1}]H^{*,*}(X; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau, \tau^{-1}] \otimes H^*(X(\mathbb{C}); \mathbb{Z}/p)$. Therefore we have the isomorphism of spectral sequences

$$[\tau^{-1}]E_2^{*,*,*}(X) \cong \mathbb{Z}/p[\tau, \tau^{-1}] \otimes E_2^{*,*}(X(\mathbb{C})).$$

□

Lemma 7.2. *Let $h^*(-)$ be mod p-theory and $dim(X) = d$. If $n > d + m/2$, then $h^{m,n}(X) \cong h^m(X(\mathbb{C}))$.*

Proof. Let $x \in H^{m,n}(X; \mathbb{Z}/p)$. Suppose $n > d + m/2$. Then $w(x) > 2d$ so $w(d_r(x)) \geq 2d$ for the differential d_r of AHss. If $2n' - m' \geq 2d$, then $n' \geq d$ and $H^{m',n'}(X; \mathbb{Z}/p) \cong H^{m'}(X(\mathbb{C}); \mathbb{Z}/p)$. Thus we know the lemma. □

We want show the Conner-Floyd type theorem for $AP(n)^{*,*}(-)$ and $AK(n)^{*,*}(-)$.

Lemma 7.3.

$$AP(n)^{*,*}(AP(n)) \cong P(n)^*(P(n)) \otimes \mathbb{Z}/p[\tau] \cong AP(n)^{*,*} \otimes RP \otimes \Lambda(Q_0, ..., Q_{n-1}).$$

Proof. (Compare [Ya1],[Wu]) For the cofiber sequence (3.7), we get the long exact sequence

$$\xleftarrow{\delta_k} AP(n)^{*-2p^n+1,*-p^n+1}(AP(n)) \xleftarrow{v_n*} AP(n)^{*,*}(AP(n)) \longleftarrow AP(n)^{*,*}(AP(n+1)).$$

By induction, we assume the isomorphism in the lemma for n. Let $\iota \in AP(k)^{0,0}(AP(k))$ represents the identity map of $AP(n)$. Then $v_n^*\iota = v_n\iota$. Let $x = aP^IQ_J\iota \in AP(n)^{*,*}(AP(k))$ for $a = \tau^s$. Then we see

$$v_n^*P^IQ_J(\iota) = P^IQ_J(v_n\iota) = v_n(P^IQ_J\iota) \quad mod(p, .., v_{n-1}).$$

The last equation is shown by the Cartan formula in $P(n)^*(-)$ and $Q_j(v_n) = 0$ and $P^I(v_n) \in Ideal(p, ..., v_{n-1})$. Thus we get $v_n^* x = v_n x$.

Hence we have $AP(n)^{*,*}(AP(n+1)) \cong AP(n)^{*,*}(AP(n))/(v_n)$.

Next we consider the Sullivan exact sequence

$$AP(n)^{*+2p^n-1,*+2p^n-1}(AP(n+1)) \xrightarrow{v_n} AP(n)^{*,*}(AP(n)) \longrightarrow AP(n+1)^{*,*}(AP(n+1)) \xrightarrow{\delta_n} .$$

Since $v_n = 0$, we get the isomorphism

$$AP(n+1)^{*,*}(AP(n+1)) \cong AP(n)^{*,*}(AP(n))/(v_n) \otimes \Lambda(Q'_n).$$

By induction, we get the lemma. $\square$

Remark From the above lemma, note that $AP(n)^{*,*}(X)$ theory is a $P(n)^*$-module. In geneal, we do not prove that $Ah^{*,*}(-)$ is a h^*-module, while it is a MU^*-module.

Similarly, we get

$$AP^{*,*}(AP(n)^{\wedge s}) \cong P(n)^*(P(n)^{\wedge s}) \otimes \mathbb{Z}/p[\tau] \quad for\ s \geq 1.$$

Notice that the realization map $t_{\mathbb{C}}^{m,m'}$ is injective for each m, m'. (Injective for homogeneous elements.)

Theorem 7.4. *The theory $AP(n)^{*,*}(-)$ has the natural product μ which is associative and commutaive for odd prime p. (When $p = 2$, $\mu - \mu t = v_n Q_{n-1} Q_{n-1} \tau$ for the twisted map t.) Moreover $AP(n)^{*,*}(-)$ is a $BP^*(BP)$-module and $Q_i, 0 \leq i \leq n-1$ generates the exterior algebra.*

Proof. This is an analogue of arguments of Würgler [Wu]. We will prove the commutaivity. We get the product map

$$\mu : AP(n) \wedge AP(n) \to AP(n)$$

as an element representing $1 \in AP(n)^{0,0}(AP(n) \wedge AP(n))$. For $p : odd$, we still know $P(n)^*(-)$ is commutaive. Hence $t_{\mathbb{C}}(\mu - \mu t) = 0$. This shows $\mu - \mu t = 0$ also in $AP(n)^{*,*}(-)$ theory by the injectivity of $t_{\mathbb{C}}^{0,0}$. Since $t_{\mathbb{C}}(\mu - \mu t) = v_n Q_{n-1} \otimes Q_{n-1}$ for $p = 2$, we get the resuts for $p = 2$. The other properties also hold since so does in $P(n)^*(-)$ theory and the injectivity of $t_{\mathbb{C}}^{m,m'}$. $\square$

Corollary 7.5. *Let $AK(n)^{*,*}(-) = [v_n^{-1}]Ak(n)^{*,*}(-)$ be the motivic Morava K-theory. Then we get the isomorphism*

$$AK(n)^{*,*}(X) \cong K(n)^* \otimes_{BP^*} AP(n)^{*,*}(X).$$

Proof. Since $AP(n)^{*,*}(AP(n))$ is a $BP^*(BP)$-module, so is the AHss $E_r^{*,*,*}$ converging $AP(n)^{*,*}(X)$. ¿From the exact functor theorem, we know

$$E_r^{*,*,*} \mapsto E_r^{*,*,*} \otimes_{P(n)^*} K(n)^*.$$

is the exact functor. Hence

$$H(E_r^{*,*,*}, d_r) \otimes_{P(n)^*} K(n)^* \cong H(E_r^{*,*,*} \otimes_{P(n)^*} K(n)^*, d_r \otimes 1)$$

Of couse the lefthand side is $E_{r+1}^{*,*,*} \otimes_{P(n)^*} K(n)^*$. By induction, we can show that the righthand side is isomorphic to the $r+1$-th term of AHss converging to $AK(n)^*(X)$ since

$$E_2^{*,*,*} \otimes_{P(n)^*} K(n)^* \cong H^{*,*}(X; \mathbb{Z}/p) \otimes K(n)^*$$

which is the E_2-term of AHss converging to $AK(n)^{*,*}(X)$. □

Here we give a good example ; the case $X = B(\mathbb{Z}/p)^m$. We still know that the fibering given in §2 (2.12) implies the isomorphisms

$$AP(1)^*(X) \cong BP^*(X) \otimes \mathbb{Z}/p[\tau] \cong BP^*[y_1, ..., y_n]/([p](y_1), ..., [p](y_n)) \otimes \mathbb{Z}/p[\tau].$$

On the otherhand, there is a decomposition [Ya4]

$$H^{*,*}(X; \mathbb{Z}/p) \cong \oplus \Lambda(Q_1, ..., Q_n)G_n.$$

with $w(G_n) = n$. The differntial of spectral sequence for $P(1) = BP/p$-theory are just $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$ and we see that $grP(1)^*(X) \cong \oplus P(n+1)^* Q_0 ... Q_n G_n$. Hence we get $E(AP(1))_2^{*,*,*} \cong E(P(1))_2^{*,*} \otimes \mathbb{Z}/p[\tau]$ from the naturality of the realization map $t_{\mathbb{C}}$. Thus we get also $AP(1)^{*,*}(X) \cong BP^*(X) \otimes \mathbb{Z}/p[\tau]$.

We aslo have the fiollowing lemma.

Lemma 7.6. *If $\sum v_n y_n = 0 \in ABP^*(X)$, then there is $x \in H^{*,*}(X; \mathbb{Z}/p)$ such that $Q_n(x) = \rho(y_n)$ where $\rho : ABP \to AH\mathbb{Z}/p$ is the natural (Thom) map.*

Proof. It is just the motivic version of the arguments of Tamanoi [Ta]. (See [Ta] for details.) □

8. ALGEBRAIC COBORDISM

By extending Quillen's [Qu] arguments, Levine and Morel defined the algebraic cobordism theory $\Omega^*(-)$ as the universal theory in theories having transfers and Chern classes [Le-Mo 1,2] (We say that $h^*(X)$ is a theory having transfers and Chern classes if this theory satifies the actioms A1 to A4 in [Le-Mo1]). We note that $\Omega^*(-)$ is not a cohomology theory. The ring $\Omega^*(X)$ is constructed as

$$\Omega^*(X) = \{[f : M \to X]\}/(relations).$$

Here f is a map from a smooth variety M to X of pure codimension, namely, $dim_{f(y)}(X) - dim_y(M)$ is constant for all y in the same connected component of M. Relations are given so that we can define Chern classes or formal group laws (for details, see [L-M 1]). Given theory $h^*(-)$ having transfers and Chern classes, the map

$$(8.1) \quad \rho_h : \Omega^*(-) \to h^*(-)$$

is defined by $\rho_h([f : M \to X]) = f_*(1_M)$ where $1_M \in h^0(M)$ represents the identity element.

We have commutaive diagram

$$\begin{array}{ccccc} \Omega^*(X) & \xrightarrow{\rho_{MGL}} & MGL^{2*,*}(X) & \xrightarrow{t_{MU}} & MU^*(X(\mathbb{C})) \\ {\scriptstyle =}\downarrow & & {\scriptstyle \rho^{2*,*}}\downarrow & & {\scriptstyle \rho^*}\downarrow \\ \Omega^*(X) & \xrightarrow{\rho_{CH}} & CH^*(X) \cong H^{2*,*}(X) & \xrightarrow{t_H} & H^*(X(\mathbb{C})) \end{array}$$

Levine and Morel [Le-Mo2] proves that

$$(8.2) \quad \Omega^*(pt) \overset{t_{MU}\rho_{MGL}}{\cong} MU^*(pt), \quad \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \overset{\rho_{CH} \otimes_{\Omega^*} Z}{\cong} CH^*(X).$$

Moreover they conjecture that ρ_{MGL} are always isomorphisms.

Let $K^0(X)$ be the Grothendieck group of algebraic vector bundles over X. Let $\tilde{K}(1)^*(X)$ is the integral K-theory, that is, $\tilde{K}(1)^* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$. Then Levine Morel also proves that

$$(8.3) \quad K^0(X) \otimes \tilde{K}(1)^* \cong \Omega^*(X) \otimes_{\Omega^*} \tilde{K}(1)^*.$$

Giving a sequence $\alpha = (a_1, a_2, ..), a_i \geq 0$, recall the Landweber-Novikov operation is defined by

$$S_\alpha(f_*(1_M)) = f_*(c_\alpha(N_f)) \quad for \ [f : M \to X] \in \Omega^*(X)$$

where $c_\alpha(N_f)$ is the Chern class of the normal bundle of $f(M)$ in X so that S_α in $MU^*(X)$ is the usual Landweber-Novikov operation. As the arguments in Section 4, we get the multiplicative projection $\Phi : \Omega^*(-)_{(p)} \to \Omega^*(-)_{(p)}$ such that $\Phi(x_i) = x_i$ if $i = p^j - 1$ for some j otherwise $\Phi(x_i) = 0$. Define the algebraic Brown-Peterson theory

$$\Omega^*_{BP}(X) = Im(\Phi(\Omega^*(X)_{(p)}) \subset \Omega^*(X)_{(p)}.$$

Hence if $h^*(-)$ is a theory having transfers and Chern classes, then there is the natural map $\rho_{BP,h} : \Omega^*_{BP}(X) \to h^*(X)_{(p)}$ compatible with $\rho_{h(p)}$.

Lemma 8.1. *Identifying $\Omega^* = MU^*$, we get $\Omega^*_{BP}(X) \cong BP^* \otimes_{\Omega^*_{(p)}} \Omega^*(X)_{(p)}$.*

9. Classifying spaces

Let G be an algebraic group over k. Consider a representation $G \to GL_n(k)$ and S be a closed set such that G acts freely on $\mathbb{A}^n - S$. Define the classifying space BG by

$$(9.1) \quad BG = Colim_{\{N \to \infty, \ codim(S) \to \infty\}} (\mathbb{A}^n - S)/G.$$

We study the motivic cobordisms of BG when $k = \mathbb{C}$.

Hereafter, we always assume $k = \mathbb{C}$.

In particular we consider the groups which satisfy

$$(9.2) \quad \Omega^*_{BP}(BG) \cong ABP^{2*,*}(BG) \cong BP^*(BG).$$

Let G be a connected group and T its maximal torus. Suppose that $H^*(BG)$ has no p-torsion. Then $BP^*(BG) \cong BP^* \otimes H^*(BG)_{(p)}$. Moreover we see

$$H^*(BG)_{(p)} \cong H^*(BN_G(T))_{(p)} \cong H^*(BT)^{W_G(T)}_{(p)}.$$

The righthand side ring is the image of the transfer $Tr(-) = Cor_T^{N_G(T)}(-)$. Chow rings also have the transfer, we have

$$CH^*(BN_G(T))_{(p)} \cong Tr(CH^*(BT))_{(p)} \cong Tr(H^*(BT))_{(p)} \cong H^*(BN_G(T))_{(p)}.$$

Totaro (see [Ve1] for details) defined the Gottlieb type transfer

$$Gtr : CH^*(BN_G(T)) \to CH^*(BG)$$

such that $Gtr.i^* = \chi(G/N_G(T))id_G = id_G \ mod(p)$ where $\chi(-)$ is the Euler characteristic and $i^* : CH^*(G) \to CH^*(BN_G(T))$ is the restriction map. Hence we have

$$CH^*(BG)_{(p)} \cong Gtr(CH^*(BN_G(T)_{(p)}) \cong Gtr(H^*(BN_G(T))_{(p)} \cong H^*(BG)_{(p)}.$$

Since $\Omega^*_{BP}(X) \otimes_{BP^*} Z_{(p)} \cong CH^*(X)_{(p)}$, we have

Proposition 9.1. *Let G be a connected group such that $H^*(BG)$ has no p-torsion. Then* (9.2) *holds.*

Examples ; $\Omega^*_{BP}(BU(n)) \cong BP^*[c_1,...,c_n]$, $\Omega^*_{BP}(BSp_{2n}) \cong BP^*[c_2,...,c_{2n}]$ where c_i are the Chern classes for the natural representations.

By S.Wilson [Wi], we know

$$BP^*(BO(n)) \cong BP^*[c_1,...,c_n]/(c_{odd} = \bar{c}_{odd}).$$

where $\bar{c}_i$ is the complex conjugation of c_i (c_i is the i-th Chern class of the complexication of the universal real bundle.) Totaro showed [To2] $BP^*(BO(n))\otimes_{BP^*}\mathbb{Z}_{(2)} \cong CH^*(BO(n))_{(2)}$. Hence we also know

Proposition 9.2. *Let $G = O(n)$ and $p = 2$. Then* (9.2) *holds.*

Next consider finite abelian p-groups. Let us study the case $G = \mathbb{Z}/p^s$. For a smooth X, the cofibering given in §2 induces the maps

$$(**) \quad \Omega^*_{BP}(X \times B\mathbb{Z}/p^s) \leftarrow \Omega^*_{BP}(X \times B\mathbb{G}_m) \xleftarrow{[p^s]} \Omega^*_{BP}(X \times B\mathbb{G}_m)$$

where $\Omega^*_{BP}(X \times B\mathbb{G}_m) \cong \Omega^*_{BP}(X) \otimes_{\Omega^*_{BP}} \Omega^*_{BP}(B\mathbb{G}_m) \cong \Omega^*_{BP}(X)[y]$ for $deg(y) = (2,1)$. Note that the above sequence $(**)$ is not neccessary exact but the composition is zero map. But we already know that $CH^*(X \times B\mathbb{Z}/p^s) \cong CH^*(X)[y]/(p^s)$ and $BP(X \times B\mathbb{Z}/p^s) \cong BP(X)[y]/([p^s](y))$. Hence we get

$$\Omega^*_{BP}(X \times B\mathbb{Z}/p^s) \cong \Omega^*_{BP}(X)[y]/([p^s](y)).$$

Similarly, we see that $ABP^{2*,*}(X \times B\mathbb{Z}/p^s)$ is isomorphic to the above ring.

Proposition 9.3. *If G has a Sylow p-subgroup isomorphic to an abelian p-group, then* (9.2) *holds.*

Remark. For the elementary abelian p-group cases are still studied in the preceding section. For $P(1)$-theory, we have $AP(1)^{*,*}(BG) \cong BP^*(BG) \otimes \mathbb{Z}/p[\tau]$. When $s > 1$, of course, there are nonzero differential not of the form $d_{p^i-1} = v_i \otimes Q_i(x)$. Let $G = \mathbb{Z}/p^r$ and consider AHss for $AP(1)^{*,*}(BG)$. Then the only nonzero differential is

$$d_{2p^r-1}(x) = v_1^{1+p+...p^{r-1}} y^{p^r}.$$

since $[p^r](y) = v_1^{1+p+...p^{r-1}} y^{p^r} \ mod(p)$.

For nonabelian p-groups, we give here partial results only. It seems quite difficult to know $CH^*(BG)$ exactly, in general.

Recall the Totaro's cycle map

$$\bar{cl} : CH^*(X)_{(p)} \cong ABP^{2*,*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \xrightarrow{t_{\mathbb{C}}^{2*,*}} BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}_{(p)}.$$

We will study the groups such that

$$(9.3) \quad ABP^{2*,*}(BG)/(BP^*Ker(\bar{cl})) \cong BP^*(BG).$$

Typical examples are $G = p_+^{1+2}$ the extraspecial p-group of order p^3 of exponent p for odd prime p (when $p = 2$, $2_+^{1+2} = D_8$; the dihedral group of order 8), or $G = PGL_3$; the projective linear group for $p = 3$ (see [Ve1]). Both cases hold that $Q_1H^{even}(BG) = 0$ and $Q_1|H^{odd}(BG)$ is injective. Hence the BP-theory is given by

$$grBP^*(BG) \cong BP^* \otimes H^{even}(BG)/(v_1Q_1H^{odd}(BG)).$$

Proposition 9.4. *If $G = p_+^{1+2}$ for all primes or PGL_3 for $p = 3$, then* (9.3) *holds.*

Proof. We prove the theorem for $G = p_+^{1+2}$ and p:odd. The other groups are also proved similarly. For this case, there is the decomposition [Ya4]

$$\tilde{H}^*(BG) \cong \mathbb{Z}/p\{G_0\} \oplus \mathbb{Z}/p^2\{G_0'\} \oplus \Lambda(Q_1)G_1$$

with G_0, G_0', Q_1G_1 are in the image $t_{\mathbb{C}}(CH^*(BG))$. Hence

$$grBP^*(BG) \cong BP^*/p\{G_0\} \oplus BP^*/p^2\{G_0'\} \oplus P(2)^*\{Q_1G_1\}.$$

Here G_1 is generated as a $t_{\mathbb{C}}(CH^*(BG))$-module by 3-dimensional two elemenets a_1, a_2. Since it is of course torsion element, we can take $a_1, a_2 \in H^{3,2}(BG)$ that is $w(a_i) = 1$. Hence we get

$$H^{*,*}(BG;\mathbb{Z}) \supset \mathbb{Z}/p\{G_0\} \oplus \mathbb{Z}/p^2\{G_0'\} \oplus \Lambda(Q_1)G_1$$

with $w(G_1) = 1$ $w(G_0) = w(G') = 0$. From Corollary 5.2 and AHss, we have the proposition. □

Here we recall some usefull fact about multiplying τ. Write $H^i(X, H^j_{\mathbb{Z}/2})$ the Zarisky cohomology of X with the coefficient in presheaf $H^j_{et}(V;\mathbb{Z}/2)$ for open subset V of X. From the result of Voevodsky we have the long exact sequence (Lemma 2.4 in [Or-Vi-Vo])

$$H^{m,n-1}(X;\mathbb{Z}/2) \xrightarrow{\tau} H^{m,n}(X;\mathbb{Z}/2) \to H^{m-n}(X;H^n_{\mathbb{Z}/2}) \to H^{m+1,n-1}(X;\mathbb{Z}/2).$$

In particular we get

Lemma 9.5. *(Lemma 2.4 in [Or-Vi-Vo]) Let X be smooth. Then $\tau : H^{n,n-1}(X;\mathbb{Z}/2) \to H^{n,n}(X;\mathbb{Z}/2)$ is injective.*

Remark If $B(m,p)$ condition is satisfied, then the similar fact holds for odd primes p. This is also explained by the Bloch-Ogus spectral sequence

$$E_2^{i,j} \cong H^i(X;H^j_{\mathbb{Z}/2}) \Longrightarrow H^{i+j}_{et}(X;\mathbb{Z}/2)$$

where $E_2^{i,j} = 0$ unless $0 \le i \le j$.(Theorem 1.3 in [To3]).

Let G be a simply connected Lie group. Then $H^3(G;Z) \cong \mathbb{Z}$ and $H^4(G;Z) \cong 0$. Suppose that $H^*(G;Z)$ has p-torsion. Then it is known that there is an element $x' \in H^3(G;\mathbb{Z})$ such that $0 \ne Q_1x' \in H^{2p+2}(G;\mathbb{Z}/p)$. Taking classifying space, we get the element $x \in H^4(BG;\mathbb{Z})$ such that $Q_1x \ne 0$ in $H^{2p+3}(BG;\mathbb{Z}/p)$. By Totaro [To2] it is known that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$. Hence there is $s \ge 1$ such that $p^sx \in H^4(BG)$ is in $Im(cl)$.

Lemma 9.6. *Suppose that $B(3,p)$ holds. Let x be an element in $H^4(BG;\mathbb{Z})$ such that $px \in Im(cl)$. Then we can take $x' \in H^{4,3}(X;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x$.*

Proof. Let $\{px\} = a \in H^{4,2}(BG)$. We consider in the coefficient $\mathbb{Z}/p^2$. Let τ_{p^2} be a $\mathbb{Z}/p^2$-module generator of $H^{0,1}(pt;\mathbb{Z}/p^2)$. Then $\tau_{p^2}^2 a = px \in H^{4,4}(BG;\mathbb{Z}/p^2)$ idefining $x \in H^{4,4}(BG;\mathbb{Z}/p)$ since so in the toplogical case. But $\tau : H^{4,3}(X;\mathbb{Z}/p) \to H^{4,4}(X;\mathbb{Z}/p)$ injective from Lemma 9.5. This means $\tau a = 0 \in H^{4,3}(X;\mathbb{Z}/p)$. Hence there is $x' \in H^{4,3}(X;\mathbb{Z}/p^2)$ so that $\tau_{p^2}a = px'$. We get $t_{\mathbb{C}}(x') = x$ since $\tau_{p^2}(px') = px$. □

When $G = SO(4), G_2, Spin(7)$ for $p = 2$ or $G = F_4$ for $p = 3$, the assumption of the above lemma hold, in fact, we can take $px = c_2$ the second Chern class of some representation. Hence we can identify $x \in H^{4,3}(BG;\mathbb{Z}/p)$.

Remark The submodules in Proposition 5.7 in [Ya4] are just $Ih^{*,*}(BSO(4))$ and $h^{*,*}(BSO(4)$.

We consider the case $X = BSO(4), G_2, Spin(7)$ and $p = 2$. The mod 2-cohomology of $BSO(4)$ is $H^*(BSO(4);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2,w_3,w_4]$. The cohomology operations are given by

$$Q_0w_2 = w_3,\ Q_1w_3 = w_3^2,\ Q_1w_4 = w_4w_3,\ Q_1Q_2w_4 = w_3^2w_4^2.$$

The integral cohomology is written as

$$H^*(BSO(4))_{(2)} \cong \mathbb{Z}_{(2)}[w_2^2, w_4] \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_3]\{w_3\}).$$

In AHss converging to $BP^*(BSO(4))$, non-zero differentials are $d_{2^{i+1}-1}(x) = v_i \otimes Q_i(x)$ for $i = 1, 2$. We can compute (see [Ya4])

$$E_\infty^{*,*} \cong E_8^{*,*} \cong Z_{(2)}[c_2] \otimes (BP^*[c_4]\{1, 2w_4\} \oplus P(2)^*[c_3]\{c_3\} \oplus P(3)^*[c_3, c_4]\{c_3c_4\}).$$

We know that the element corresponding $2w_4$ is represented by a Chern class c_2' of some representation and this means the Totaro's cycle map $\tilde{cl}$ is epic. Indeed, Totaro and Padharipande [Pa] shows that this map is isomorphic, namely,

$$CH^*(BSO(4))_{(2)} \cong Z_{(2)}[c_2, c_3, c_4, c_2']/(2c_3, c_3c_2', {c_2'}^2 - 4c_4).$$

Proposition 9.7. $ABP^{2*,*}(BSO(4)) \cong BP^*(BSO(4))$.

Proof. By Corollary 9.6, we can take $w_4 \in H^{4,3}(BSO(4);\mathbb{Z}/2)$. ¿From the naturality of AHss for the realization map, we get the proposition. □

The cases $X = BG_2, BSpin(7)$ are quite similar to the case $X = BSO(4)$. The cohomology $H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7]$ and

$$grBP^*(BG_2) \cong Z_{(2)}[c_4, c_6] \otimes (BP^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

The Chow ring of BG_2 is genrated as a ring by Chern classes c_2, c_4, c_6 and c_7, and there is an epimorphism [To2], [Ya4]

$$CH^*(BG_2)_{(2)} \to Z_{(2)}[c_2, c_4, c_6, c_7]/((c_2^2 - 4c_4), 2c_7, c_2c_7) \cong BP^*(BG_2) \otimes_{BP^*} \mathbb{Z}_{(2)}.$$

In [Ya4] we prove $2c_7 = c_2c_7 = 0$ also in $CH^*(BG_2)$.

Theorem 9.8.

$$CH^*(BG_2)_{(2)} \cong \mathbb{Z}_{(2)}[c_2, c_4, c_6, c_7]/((c_2^2 - 4c_4), 2c_7, c_2c_7)$$

Proof. We only need to prove $c_2^2 - 4c_4 = 0$. Merkurjev showed ([To2]) that the Grothendieck group $K^0(BG)$ of algebraic vector bundles is isomorphic to the usual K-theory $K(BG)$. We also recall the Conner-Floyd relation

$$\tilde{A}K(1)^{*,*}(X) \cong ABP^{*,*}(X) \otimes_{BP^*} \tilde{K}(1)^*.$$

Hence $a = (c_2^2 - 4c_4)$ must be (higher) v_1-torsion in $ABP^{*,*}(BG_2)$.

Suppose $a \neq 0$. Suppose that $d_5(b) = v_1^2 a$ in the spectral sequence converging to $ABP^{*,*}(BG_2)$. Then $b \in H^{3,2}(BG_2) \cong H_L^{3,2}(BG_2)$ by the $B(n,2)$- condition. Here b is not infinite divisible since neither its d-image $v_1^2 a$. Hence there exists b' in the usual mod 2 cohomology $H^3(BG_2;\mathbb{Z}/2)$ corresponding b. This is a contradiction.

Suppose that $d_3(b) = v_1 a$. Then there is $b \in H^{5,3}(BG_2;\mathbb{Z}/2)$ which is not infinite divisible. If b is a torsion element, then there is $b' \in H^{4,3}(BG_2;\mathbb{Z}/2)$ whose (higher) Bockstein is b. Then from Lemma 9.6, $\tau b' \neq 0$. By $B(n,2)$-condition, this means $t_{\mathbb{C}}(b') \neq 0$ in $H^4(BG_2;\mathbb{Z}/2)$. But there is not such element.

Let b be torsion free element. It is known [Vo1] that $H^{i,j}(X;\mathbb{Q}) \cong H_L^{i,j}(X;\mathbb{Q})$. Hence $\mathbb{Z} \subset H_L^{5,3}(BG_2;\mathbb{Z})$. This means $H_L^{5,3}(BG_2;\mathbb{Z}/2) \cong H^5(BG_2;\mathbb{Z}/2)$ contains $\mathbb{Z}/2$ and this is a contradiction. □

We also see that $w(w_4) = 2$ by Corollary 9.6.

Corollary 9.9. *$ABP^{2*,*}(BG_2) \cong BP^*(BG_2)$.*

The mod 2 cohomology is $H^*(BSpin(7);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8]$ and $grBP^*(BSpin(7))$ is isomorphic to

$\mathbb{Z}_{(2)}[c_4, c_6] \otimes (BP^*[c_8]\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \oplus P(3)^*[c_7]\{c_7\} \oplus P(4)^*[c_7, c_8]\{c_7c_8\})$.

Hence we know that $BP^*(BSpin(7)) \otimes_{BP^*} \mathbb{Z}$ is isomorphic to

$$\mathbb{Z}_{(2)}[c_4, c_6, c_8] \otimes (\mathbb{Z}\{1, 2w_4, 2w_8, 2w_4w_8\} \oplus \mathbb{Z}/2\{v_1w_8\} \oplus \mathbb{Z}/2[c_7]\{c_7\}).$$

It is know that elements $2w_8, 2w_4w_8$ are represented by Chern classes but v_1w_8 is not. However Totaro shows that the cycle map $\tilde{cl}$ is epic. Here we give the another proof.

Lemma 9.10. *In $BP^*(BSpin(7)) \otimes_{BP^*} \mathbb{Z}_{(2)}$, the element $\{v_1w_8\}$ is in $Im(\tilde{cl})$.*

Proof. Let $a = v_1w_8$ in $E_\infty^{*,*}$. By the Conner-Floyd type therorems for the motivic and the usual thories

$$ABP^{2*,*}(BG) \otimes_{BP^*} \tilde{K}(1)^* \cong A\tilde{K}(1)^{2*,*}(BG)$$
$$\cong \tilde{K}(1)^*(BG) \cong BP^*(BG) \otimes_{BP^*} \tilde{K}(1)^*.$$

Moreover we know $ABP^{2*,*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(X)_{(p)}$. Hence there exists element $c \in CH^*(BSpin(7))$ such that $\bar{cl}(c) = v_1^s a$ for some $s \geq 0$. By dimensional reason, $s = 0$ or 1. If $s = 1$, then $c \in CH^2(BSpin(7))$. But it is known from Totaro (Cor. 3.5 in [To 2]) that

$$CH^2(BG)_{(p)} \cong (BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)})^4.$$

Hence we see that $\bar{cl}(c) = a$. □

Proposition 9.11. *For $p = 2$ and $G = Spin(7))$, if $w(w_8) = 2$, then the isomorphisms (9.3) holds.*

Lemma 9.12. *The weght $w(w_8) = 2$ or 4 in $H^{*,*}(BSpin(7);\mathbb{Z}/2)$.*

Proof. Let $X = BSpin(7)$. Consider the AHss

$$E_2^{*,*,*} \cong H^{*,*}(X; BP^*/2) \Longrightarrow AP(1)^*(X).$$

First we think $w_8 \in H^{8,8}(X;\mathbb{Z}/p)$. Let $\xi \in H^{6,3}(X, \mathbb{Z}/2)$ be the element with $\bar{cl}(\xi) = \{v_1 \otimes w_8\}$ in $BP^*(X) \otimes_{BP^*} \mathbb{Z}/2$. For each $0 \neq a \in E_\infty^{*,*}$ in AHss converging to $Ah^{*,*}(X)$, we use the notation $\{a\}$ which is some element in $Ah^{*,*}(X)$ representing a. From Lemma 7.2, for sufficient large N we have

$$(*) \quad \tau^{N+4}\{\xi\} = \tau^N\{v_1 \otimes w_8\} \quad \text{in } AP(1)^{*,*}(X)$$

here we identify $\xi \in E_\infty^{6,3,0}, v_1 \otimes w_8 \in E_\infty^{8,8,-2}$ while $\{\xi\} \in AP(1)^{6,3}(X)$, $\{v_1 \otimes w_8\} \in AP(1)^{6,7}(X)$.

In $H^{*,*}(X;\mathbb{Z}/2)$, from Lemma 9.2, we know $\tau^s\xi = 0$ for $s = 1$ or 2. This means that there is $a \in E_\infty^{*,*,minus}$ with $\tau^s\{\xi\} = \{a\}$. By $(*)$ and dimensional reason $a = v_1 \otimes w'$. Here $t_{\mathbb{C}}(w) = w_8$ and $w(w') = 2s$.

□

10. ALGEBRAIC GROUPS CORRESPONDING LIE GROUPS

Let G be a simply connected complex Lie group and $G_{\mathbb{C}}$ the corresponding reductive algebraic group over $\mathbb{C}$. Let T be the maximal torus of G and BT its classifying space. Consider the fibering

$$G \xrightarrow{\pi} G/T \xrightarrow{i} BT.$$

Grothendieck [G] proved that

$$CH^*(G_{\mathbb{C}}) \cong H^*(G/T)/(Ideal(i^*\tilde{H}^*(BT))).$$

By using arguments of Grothendieck, we have its cobordism version.

Theorem 10.1. $MGL^{2*,*}(G_{\mathbb{C}}) \cong \Omega^*(G_{\mathbb{C}}) \cong MU^*(G/T)/(Ideal(i^*\tilde{MU}(BT)))$.

Proof. The second isomorphism is proved in [Ya4]. The first isomorphism is immediate from the second isomorphism and the Morel-Hopkins result Corollary 3.8. □

Let p be a fixed prime. From the Grothendiek result, Kac [K] noted that $CH^*(G_{\mathbb{C}})/p \cong \pi^*(H^*(G/T))/p$. Recall $\Omega^* \cong MU^* \cong \mathbb{Z}[x_1, ...]$ with $|x_i| = -2i$ and identify $v_i = x_{p^i-1}$.

Theorem 10.2. *Let $I = (p, v_1, ...)$ be the invariant ideal of MU^*. Then*

$$\Omega^*(G_{\mathbb{C}})/I^2 \cong \pi^*(MU^*(G/T))/(I^2)', \quad \textit{where } (I^2)' = I^2MU^*(G)\cap\pi^*(MU^*(G/T)).$$

By the Borel theorem, we can write $H^*(G;\mathbb{Z}/p) \cong P(y_{even})/p \otimes \Lambda(x_{odd})$ where $P(y_{even})$ is a truncated polynomial algebra of even degree generators y_{even} and $\Lambda(x_{odd})$ is the exterior algebra of odd degree generators x_{odd}. When $p = 2$ we take y_{even} as a power of some x_{odd}. (Here let $P(y_{even})$ be the plonomial over $\mathbb{Z}$ for later convenience.) Then the result of Grothendieck and Kac [K] is stated as

$$CH^*(G_{\mathbb{C}})/p \cong P(y_{even})/p.$$

Let Q_i be the Milnor primitive operation inductively defined by $Q_i = [Q_{i-1}, P^{p^{i-1}}]$ and $Q_0 = \beta$; the Bockstein operation where $P^{p^{p-1}}$ is the p^{p-1}-th reduced power operation. It is known that $Q_i(x_{odd}) \in P(y_{even})/p$ for all $i \geq 0$.

Theorem 10.3. *There is an ideal $F \subset \Omega^*(G_{\mathbb{C}})$ such that its graded algebra $gr\Omega^*(G_{\mathbb{C}}) = F \oplus \Omega(G_{\mathbb{C}})/F$ induces the isomorphism*

$$gr\Omega^*(G_{\mathbb{C}})/I^2 \cong \Omega^* \otimes P(y_{even})/(I^2, \sum_i v_iQ_i(x_{odd})).$$

When $p \neq 2$, we can take $F = 0$. For $p = 2$, we can take F so that

$$F = Ideal(y_i^{2^{k_i}}|1 \leq i \leq s) \quad \textit{when } P(y_{even}) = \otimes_{i=1}^s \mathbb{Z}[y_i]/(y_i^{2^{k_i}}).$$

Consider the spectral sequence induced from the above (topological) fibering

$$E_2^{*,*} = H^*(BT; H^*(G;\mathbb{Z}/p)) \Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

The cohomology of the classifying space of the torus is $H^*(BT) \cong \mathbb{Z}[t_1, ..., t_l]$ with $|t_i| = 2$ where l is also the number of the odd degree generatos x_k in $H^*(G;\mathbb{Z}/p)$. It is known that there is a regular sequence $(b_1, ..., b_l)$ in $H^*(BT)/p$ such that $d_{|x_i|+1}(x_i) = b_i$. Thus we get

$$H^*(G/T;\mathbb{Z}/p) \cong P(y) \otimes \mathbb{Z}/p[t_1, ..., t_l]/(b_1, ..., b_l).$$

Theorem 10.4. *([Ya5]) Let G be a simply connected Lie group. Giving bidegree to $H^*(G;\mathbb{Z}/p)$ by $w(y_i)=0$ and $w(x_i)=1$, we have the injection*

$$P(y)/p\otimes\Lambda(x_1,...,x_l)\otimes\mathbb{Z}/p[\tau]\subset H^{*,*}(G_{\mathbb{C}};\mathbb{Z}/p)$$

such that for $p=odd$, it is a ring monomorphism, and for $p=2$, it is a ring monomorphism to $grH^{*,*}(G_{\mathbb{C}};\mathbb{Z}/2)$ and $x_i^2-y_i\tau\in Ker(t_{\mathbb{Z}/p})$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}};\mathbb{Z}/p) & \longleftarrow & H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}},G_{\mathbb{C}};\mathbb{Z}/p) & \xleftarrow{\delta} & H^{2*-1,*}(G_{\mathbb{C}};\mathbb{Z}/p)\\
{\scriptstyle t_{\mathbb{Z}/p},\cong}\downarrow & & {\scriptstyle t_{\mathbb{Z}/p}}\downarrow & & {\scriptstyle t_{\mathbb{Z}/p}}\downarrow\\
H^{2*}(G/T;\mathbb{Z}/p) & \longleftarrow & H^{2*}(G/T,G;\mathbb{Z}/p) & \xleftarrow{\delta} & H^{2*-1}(G;\mathbb{Z}/p)
\end{array}$$

where rows are exact. First note that

$$H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}};\mathbb{Z}/p)\cong CH^*(G_{\mathbb{C}}/T_{\mathbb{C}})/p\cong H^{2*}(G/T;\mathbb{Z}/p)$$

since $G_{\mathbb{C}}/T_{\mathbb{C}}$ has a cellular decomposition.

Recall that $E_2^{*,*}$ is the spectral sequence converging to $H^*(G/T;\mathbb{Z}/p)$. Since the differential in $E_r^{*,*}$ are given $d_{|b_i|}(x_i)=b_i$, we easily seen from the definition of differential, $\delta(x_i)=b_i\ mod(E_\infty^{|b_i|+1,*})$ in $H^*(G/T,G;\mathbb{Z}/p)$. Hence $b_i\neq 0\in H^*(G/T,G;\mathbb{Z}/p)$. Here note that $t_1,...,t_l$ are in $H^*(G/T,G;\mathbb{Z}/p)$, while y_i is not in $H^*(G/T,G;\mathbb{Z}/p)$ because $t_i=0\in H^*(G;\mathbb{Z}/p)$.

Similarly $t_1,...,t_l$ are in $H^{2*,*}(G/T,G_{\mathbb{C}};\mathbb{Z}/p)$, since the corresponding elements in $H^{2*,*}(G/T;\mathbb{Z}/p)$ goes to zero in $H^{2*,*}(G_{\mathbb{C}};\mathbb{Z}/p)=CH^*(G_{\mathbb{C}})/p$. Moreover $b_i\neq 0$ in $H^{2*,*}(G/T,G_{\mathbb{C}};\mathbb{Z}/p)$ since it is nonzero in $H^*(G/T,G;\mathbb{Z}/p)$. Hence there is the element $x_i\in H^{2*-1,*}(G_{\mathbb{C}};\mathbb{Z}/p)$ such that $\delta(x_i)=b_i$.

Since $t_{\mathbb{Z}/p}(\tau)=1$, we also see $t_{\mathbb{Z}/p}(y_i\tau)=t_{\mathbb{Z}/p}(x_i^2)$ for $p=2$. $\square$

Other types of motivic cohomology seem quite complicated and we consider only group G in Case I , that is, (G,p) are the exceptional Lie groups $(G_2,2)$, $(F_4,2)$,$(E_6,2)$, $(F_4,3)$, $(E_6,3)$, $(E_7,3)$ and $(E_8,5)$. For thse cases, the ordinary *mod* p-cohomology is written (see [Ya2])

$$H^*(G;\mathbb{Z}/p)\cong\mathbb{Z}/p[y]/(y^p)\otimes\Lambda(x_1,x_2,...,x_l)$$

where $|x_1|=3, |x_2|=2p+1, |y|=2p+2,\ Q_1x_1=Q_0x_2=y$. In this case $P(1)^*(G)=BP^*(G;\mathbb{Z}/p)$ is known. Consider AHss

$$E_2^{*,*}=H^*(G;P(1)^*)\Longrightarrow P(1)^*(G).$$

Th only nonzero differential is $d_{2p-1}(x)=v_1\otimes Q_1(x)$. Thus we can prove [Ya2]

$$P(1)^*(G)\cong(P(1)^*[y]/(y^p,v_1y)\oplus P(1)^*\{xy^{p-1}\})\otimes\Lambda(x_2,...,x_l)$$

Theorem 10.5. *([Ya5]) For the group (G,p) in Case I, we have the isomorphism*

$$ABP^{2*,*}(G_{\mathbb{C}})\cong\Omega^*_{BP}(G_{\mathbb{C}})\cong BP^*[y]/(py,v_1y,y^p).$$

Theorem 10.6. *Let G be a simple Lie group in Case I. Giving the bidegree to $AP(1)^{*,*}(G)$ by $w(y)=0$, $w(x)=1$ for $x=x_i,x_1y^{p-1}$, we have the $AP(1)^{*,*}$-algebra injection*

$$(P(2)^{*,*}[y]/(y^p)\oplus P(1)^*\{x_1y^{p-1}\})\otimes\Lambda(x_2,...,x_l)\otimes\mathbb{Z}/p[\tau]\subset AP(1)^{*,*}(G_{\mathbb{C}}).$$

Proof. Consider AHss

$$E_2^{*,*,*} = H^{*,*}(G_{\mathbb{C}}; P(1)^*) \Longrightarrow AP(1)^{*,*}(G_{\mathbb{C}}).$$

Then $E_2^{*,*,*} \supset H^*(G;\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \otimes P(1)^*$ with weight $w(y) = 0$ and $w(x_i) = 1$. The first differntial is $d_{2p-1} = v_1 \otimes Q_1$ and $E_{2p}^{*,*,*}$ contains the $AP(1)^{*,*}$-algebra in the theorem.

Let $x = x_1 y^{p-1}$ or x_i for $i \geq 2$. Suppose that $d_r(x) \neq 0$ for $r \geq 2p$. Since $w(d_r(x)) = 0$, we know $d_r(x) \in P(2)^*[y]/(y^p)$. But y^j is $P(2)^*$-free from Theorem 10.4 and this is a contradiction. Thus x are permanent cycles. Moreover $P(1)^*(G)$ contains $P(1)^* \otimes \Lambda(x_1 y^{p-1}, x_2, ..., x_\ell)$. Thus we get the theorem. □

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