A NOTE ON FUNCTIONAL EQUATIONS OF l-ADIC POLYLOGARITHMS

ZDZISLAW WOJTKOWIAK

CONTENTS

1. Introduction .............................. 1
2. Functional equations ................. 3
3. l-adic polylogarithms and Soulé classes 5
References .................................. 8

Abstract. We are studying l-adic analogues of the classical polylogarithms. Using an explicit arithmetic formula we show that they satisfy the same distribution equations as classical polylogarithms. We also identify l-adic polylogarithms evaluated at roots of unity with Soulé classes.

1. INTRODUCTION

In [6] we defined l-adic polylogarithms and we showed that they satisfy functional equations analogous to functional equations of classical polylogarithms. In [2], motivated by an old result of O.Gabber, we have given in a joint work with H.Nakamura an explicit formula for l-adic polylogarithms. In this note we shall use this explicit formula to get functional equations of l-adic polylogarithms. We use them in [6] to study the action of Galois groups on \( \pi_1(\mathbb{P}^1_K \setminus \{0, \mu_n, \infty\}, \mathfrak{O}) \).

Let \( K \) be a number field. We denote by \( G_K \) the Galois group \( \text{Gal}(\overline{K}/K) \). Let us fix a prime number \( l \). We choose in \( \overline{K} \) a compatible family \( (\xi_{i,j})_{n \in \mathbb{N}} \) of \( l^n \)-th roots of 1 such that \( (\xi_{i,j+1})^l = \xi_{i,j} \).

For any \( z \in \mathbb{K} \setminus \{0\} \) we choose in \( \overline{K} \) a compatible family \( (z^{\frac{1}{l^n}})_{n \in \mathbb{N}} \) of \( l^n \)-th roots of \( z \) such that \( (z^{\frac{1}{l^n+1}})^l = z^{\frac{1}{l^n}} \).

For any \( z \in \mathbb{K} \setminus \{0\} \) we define a Kummer character

\[ \kappa_z : G_K \to \mathbb{Z}_l \]

setting \( \sigma(z^{\frac{1}{l^n}}) = (\xi_{i,j})^{\kappa_z(\sigma)} \cdot z^{\frac{1}{l^n}} \) for all \( n \).

For any natural number \( n \) and any \( i \in \mathbb{Z} \) such that \( 0 \leq i < l^n \) we choose a compatible family \( ((1 - \xi_{i,j}^{\frac{1}{l^n}}) \cdot z^{\frac{x}{l^n}})_{m \in \mathbb{N}} \) of \( l^m \)-th roots of \( 1 - \xi_{i,j}^{\frac{1}{l^n}} \cdot z^{\frac{x}{l^n}} \). We assume that for any two natural numbers \( n_1 \) and \( n_2 \) such that \( n_1 > n_2 \) and for any natural number \( m \)

\[ \prod_{i \equiv i_0(l^{n_2})} (1 - \xi_{i,j}^{\frac{1}{l^{n_1}}})^{\frac{1}{l^m}} = (1 - \xi_{i,j}^{\frac{1}{l^{n_2}}})^{\frac{1}{l^m}}. \] (1.1)

For any natural numbers \( n \) and \( m \) and any \( i \in \mathbb{Z} \) such that \( 0 \leq i < l^n \) we define functions

\[ \kappa_{i,n,m}^{n,m}(z) : G_K \to \mathbb{Z}/l^m \mathbb{Z} \]

by the following equality

\[ \sigma((1 - \xi_{i,j}^{\frac{1}{l^n}}(i - \kappa_z(\sigma)) \cdot z^{\frac{x}{l^n}})^{\frac{1}{l^m}}) = \xi_{i,j}^{\kappa_{i,n,m}^{n,m}(z)(\sigma)}, \]
where $\chi : G_K \to \mathbb{Z}_l^*$ is the cyclotomic character.

The family of functions $(\kappa_i^{n,m}(z)(\sigma))_{m \in \mathbb{N}}$ is compatible, hence we get a function

$$\kappa_i^n(z) : G_K \to \mathbb{Z}_l.$$  

Let us fix $\sigma \in G_K$. Then for each $n$ we have a function

$$\mathbb{Z}/l^n \to \mathbb{Z}_l, \quad i \mapsto \kappa_i^n(z)(\sigma).$$

Let $n_1 > n_2$ and let $0 < i_0 < l^{n_2}$. The identities

$$\prod_{i \equiv i_0(l^{n_2})} (1 - \xi_{l,i_1}^{\lambda_i^{-1}}(i - \kappa_i(\sigma)) \cdot z^{\frac{1}{l^{n_2}}}) \equiv (1 - \xi_{l,i_2}^{\lambda_i^{-1}}(i_0 - \kappa_i(\sigma)) \cdot z^{\frac{1}{l^{n_2}}})$$

and

$$\prod_{i \equiv i_0(l^{n_2})} (1 - \xi_{l,i_1}^{\lambda_i^{-1}} \cdot z^{\frac{1}{l^{n_2}}}) \equiv (1 - \xi_{l,i_2}^{\lambda_i^{-1}} \cdot z^{\frac{1}{l^{n_2}}}),$$

which follow from (1.1), imply that

$$\sum_{i \equiv i_0(l^{n_2})} \kappa_i^{p+1}(z)(\sigma) = \kappa_i^{n_2}(z)(\sigma).$$

Hence we get a measure

$$\kappa(z)(\sigma) := (\kappa_i^n(z)(\sigma))_{n \in \mathbb{N}, 0 \leq i < l^n}$$

on $\mathbb{Z}_l = \lim \mathbb{Z}/l^n$. Let $n > m$ and let $p$ be a natural number or $0$. The equality (1.2) implies that

$$\sum_{i=0}^{l^{n-1}} i^p \kappa_i^n(z)(\sigma) \equiv \sum_{i=0}^{l^{m-1}} i^p \kappa_i^m(z)(\sigma) \mod l^m. \quad (1.3)$$

Hence by the definition of the integration on $\mathbb{Z}_l$ we get

$$\int_{\mathbb{Z}_l} x^p d\kappa(z)(\sigma) \equiv \sum_{i=0}^{l^{n-1}} i^p \kappa_i^n(z)(\sigma) \mod l^n. \quad (1.4)$$

It follows from (1.4) and the fact that $\kappa_i^n(z)(\sigma) \equiv \kappa_i^{n,n}(z)(\sigma) \mod l^n$ that

$$\int_{\mathbb{Z}_l} x^p d\kappa(z)(\sigma) \equiv \sum_{i=0}^{l^{n-1}} i^p \kappa_i^{n,n}(z)(\sigma) \mod l^n. \quad (1.5)$$

Let us set

$$\ell_{p+1}(z)(\sigma) := \int_{\mathbb{Z}_l} x^p d\kappa(z)(\sigma).$$

We shall call the function $\ell_p(z) : G_K \to \mathbb{Z}_l$ an $l$-adic polylogarithm evaluated at $z$.

The function $\ell_n(z)$ appears naturally when studying the action of $G_K$ on the $\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \bar{0}1)$ - torsor of $l$-adic paths from $\bar{0}1$ to $z$.

We embed the group $\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \bar{0}1)$ into the $\mathbb{Q}_l$ - algebra $\mathbb{Q}_l\{\{X, Y\}\}$ of formal power series in two non-commuting variables $X$ and $Y$. The image of the loop around 0 (resp. around 1) is the power series $e^X$ (resp. $e^Y$). The embedding is continuous and multiplicative and we denote it by $k$. Let $\gamma$ be a path from $\bar{0}1$ to $z$. Then $\gamma^{-1} \cdot \sigma(\gamma)$ is a loop in $\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \bar{0}1)$. Let $x$ be a loop around 0 - one of the standard generators of $\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \bar{0}1)$. We can write

$$\log k(\gamma^{-1} \cdot \sigma(\gamma) \cdot x^{-\kappa_i(\sigma)}) = l_1(z)_\gamma(\sigma)Y + \sum_{n=2}^{\infty} l_n(z)_\gamma(\sigma)[[X, Y]X^{n-2}] + \ldots$$

The main result in [2] is that the function $l_n(z)_\gamma$ coincides with $\ell_n(z)$ for some choice of a path $\gamma$ from $\bar{0}1$ to $z$. The path $\gamma$ determines canonically choices of compatible
families of various \( l^n \)-th roots and these choices must be the same as those used in the definition of \( \ell_n(z) \).

## 2. Functional equations

In this section we show that the function \( \ell_n(z) \) satisfies certain functional equations known as distribution relations. These equations were already proved in [6] using geometric methods. They were used in [6] in the study of actions of Galois groups on \( \pi_1(\mathbb{P}^1_K \setminus \{0, \mu_n, \infty\}, 0\bar{1}) \).

### Theorem 2.1

Let \( r \) be a positive integer prime to \( l \) and let \( \xi_r \) be a primitive \( r \)-th root of 1. We assume that \( \xi_r \) and \( z \) are in \( K \). For each \( 0 \leq a < r \) we choose a compatible family \( (\xi_r^{\nu a})_{\nu \in \mathbb{N}} \) of \( l^n \)-th roots of \( \xi_r^a \) such that each \( \xi_r^{\nu a} \in \mu_r \). Then we have

\[
\ell_m(z^r) = r^{m-1} \left( \sum_{\xi_r^a = 1} \ell_m(\xi_r^a z) \right),
\]

for some convenient choices of compatible families of \( l^n \)-th roots of \( \xi_r \cdot z \) and \( z^r \) and \( l^N \)-th roots of \( 1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^r \) and \( 1 - \xi_r^a \cdot (z^r)^r \).

### Proof

Observe that \( (\xi_r^{\nu a} \cdot z^{\nu r})_{\nu \in \mathbb{N}} \) is a compatible family of \( l^n \)-th roots of \( \xi_r^a \cdot z \) and \( \prod_{a=0}^{r-1} (\xi_r^{\nu a} \cdot z^{\nu r}) = (z^r)^{r-1} \) is a compatible family of \( l^n \)-th roots of \( z^r \). We choose a compatible family \( ((1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu})_{\nu \in \mathbb{N}})_{a=0}^{r-1} \) of \( l^N \)-th roots of \( 1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^r \) satisfying (1.1). Then \( \prod_{a=0}^{r-1} (1 - \xi_r^a \cdot \xi_r^{r a} \cdot z^r) = (1 - \xi_r^a \cdot (z^r)^r) \) is a compatible family of \( l^N \)-th roots of \( 1 - \xi_r^a \cdot (z^r)^r \) satisfying (1.1). Observe that we have the identities

\[
\prod_{a=0}^{r-1} \left( \prod_{i=0}^{r-1} \left( \frac{\sigma((1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu}))(\xi_r^{\nu a} \cdot z^{r\nu})}{(1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu})(z^r)^{r-1}} \right)^{m-1} \right) =
\]

\[
\prod_{i=0}^{r-1} \left( \prod_{a=0}^{r-1} \frac{\sigma((1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu}))(\xi_r^{\nu a} \cdot z^{r\nu})}{(1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu})(z^r)^{r-1}} \right) \right)^{m-1}
\]

\[
\prod_{i=0}^{r-1} \left( \frac{\sigma((1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu}))(\xi_r^{\nu a} \cdot z^{r\nu})}{(1 - \xi_r^a \cdot \xi_r^{\nu a} \cdot z^{r\nu})(z^r)^{r-1}} \right)^{m-1}
\]

This implies

\[
r^{m-1} \sum_{a=0}^{r-1} \sum_{i=0}^{l^n-1} \kappa_i^{n, n} (\xi_r^a z) = \sum_{i=0}^{l^n-1} \kappa_i^{n, n} (z^r).
\]

The theorem now follows from the congruences (1.2) and (1.4).

In the sequel choices of various compatible families of \( l^n \)-th roots will be evident from the proofs. We shall not indicate these choices in order to shorten and lighten the paper.

To consider the case when \( l \) divides \( r \) we need the following lemma.

### Lemma 2.2

Let \( \xi_{lr} \in K \) and let \( z \in K \). We have

\[
\kappa_{\xi_{lr} z}(\sigma) = \frac{(\chi(\sigma) - 1) j}{j} + \kappa_z(\sigma).
\]
Proof. We have
\[
\frac{\sigma(\xi_{j+n}^0, z)}{\xi_{j+n}^0} = \frac{\sigma(\xi_{j+n}^0, \xi_l\sigma)}{\xi_{j+n}^0} = \xi_l^m(\sigma) = \xi_l^m(\sigma)^{-1} \cdot \xi_l^m(\sigma) = \frac{\{\chi(\sigma)^{-1}\}j \cdot \xi_l^m(\sigma)}{\chi(\sigma) + \xi_l^m(\sigma)}.
\]
\]

Theorem 2.3. Let \( \xi_l^0 \in K \) and let \( z \in K \). We have
\[
\ell_m(z^\nu) = l^\nu(m-1) \sum_{j=0}^{l^\nu-1} \ell_m(\xi_{j+l}^0 \cdot z) + \sum_{a=1}^{m-1} \binom{m-1}{a} \nu(\nu-m) \sum_{j=0}^{l^\nu-1} j^a \ell_{m-a}(\xi_{j+l}^0 \cdot z).
\]

Proof. We have
\[
l^\nu(m-1) \prod_{i=0}^{l^\nu-1} \left( \frac{\sigma((1 - \xi_l^0(\sigma-1)(i-k_{l^\nu}^l), (\xi_l^0)^{1/2}))}{(1 - \xi_l^0 \cdot (z^\nu)^{1/2})} \right)^{l^\nu-1} \prod_{0 \leq i < l^\nu, i = j + kl^\nu} \left( \frac{\sigma((1 - \xi_l^0(\sigma-1), \xi_l^0(\sigma-1)(i-k_{l^\nu}^l), (\xi_l^0)^{1/2}))}{(1 - \xi_l^0 \cdot (z^\nu)^{1/2})} \right)^{l^\nu(m-1)^{-1}} \prod_{j=0}^{l^\nu-1} \prod_{k=0}^{l^\nu-1} \left( \frac{\sigma((1 - \xi_l^0(\sigma-1), \xi_l^0(\sigma-1)(i-k_{l^\nu}^l), (\xi_l^0)^{1/2}))}{(1 - \xi_l^0 \cdot (z^\nu)^{1/2})} \right)^{l^\nu(m-1)^{-1}} \prod_{a=0}^{l^\nu-1} \prod_{j=0}^{l^\nu-1} \prod_{k=0}^{l^\nu-1} \left( \frac{\sigma((1 - \xi_l^0(\sigma-1), \xi_l^0(\sigma-1)(i-k_{l^\nu}^l), (\xi_l^0)^{1/2}))}{(1 - \xi_l^0 \cdot (z^\nu)^{1/2})} \right)^{l^\nu(m-1)^{-1}}.
\]
Hence for any \( N \) we have
\[
\sum_{i=0}^{l^\nu-1} \left[ \sum_{0 \leq j < l^\nu} \sum_{k=0}^{l^\nu-1} k_{l^\nu}^l \xi_{l^\nu}^j \cdot \xi_{l^\nu}^j(\sigma) \right] + \sum_{a=0}^{l^\nu-1} \sum_{j=0}^{l^\nu-1} \sum_{k=0}^{l^\nu-1} k_{l^\nu}^l \xi_{l^\nu}^j \cdot \xi_{l^\nu}^j(\sigma).
\]
This implies that
\[
\ell_m(z^\nu) = l^\nu(m-1) \sum_{j=0}^{l^\nu-1} \ell_m(\xi_{j+l}^0 \cdot z) + \sum_{a=1}^{m-1} \binom{m-1}{a} \nu(\nu-m) \sum_{j=0}^{l^\nu-1} j^a \ell_{m-a}(\xi_{j+l}^0 \cdot z).
\]

Now we consider the general case.

Theorem 2.4. Let \( p \) be a positive integer. Assume that \( p = r \cdot l^\nu \) where \( r \) is prime to \( l \). Assume that \( \xi_r \) and \( \xi_{l^\nu} \) are in \( K \). Then we have
\[
\ell_m(z^p) = p^{m-1} \sum_{\xi_r = 1}^{p^{m-1}} \ell_m(\xi_r \cdot z) + p^{m-1} \sum_{a=1}^{m-1} \binom{m-1}{a} \nu(\nu-m) p^{m-1-a} \sum_{k=0}^{p^{m-1-a}} j^a \ell_{m-a}(\xi_{j+l}^0 \cdot \xi_{l^\nu}^k \cdot z).
\]

Proof. The theorem follows from Theorems 2.1 and 2.3.
Corollary 2.5. Let $G_m \subset G_K$ be a subgroup on which functions $\ell_k(z)$ vanish for all $z \in K$ and all $k < m$. Then

$$\ell_m(z^p) = p^{m-1} \sum_{\xi^p = 1} \ell_m(\xi \cdot z)$$
on G_m.

3. $\ell$-adic Polylogarithms and Soulé Classes

3.1. We recall briefly the construction of the Soulé classes. Let $K$ be a number field. Let us set $K_n := K(\mu_n)$ and $K_\infty := K(\mu_\infty)$. The extension $K \subset K_n$ is unramified outside $l$. Let $\mathcal{M}$ be a maximal Galois extension of $K$ unramified outside $l$.

Let us set $G := \text{Gal}(\mathcal{M}/K)$, $G_n := \text{Gal}(\mathcal{M}/K_n)$, $G_\infty := \text{Gal}(\mathcal{M}/K_\infty)$, $\Gamma_n := \text{Gal}(K_n/K)$ and $\Gamma_\infty := \text{Gal}(K_\infty/K)$. The groups $G_n$ and $G_\infty$ are normal subgroups of $G$ and we have $G/G_n \simeq \Gamma_n$ and $G/G_\infty \simeq \Gamma_\infty$. Let us set $R_n := O_{K_n}[1/l]$ and $R := O_K[1/l]$. Let us set $X_n := \text{Spec}R_n$, $X_\infty := \lim\limits_{\rightarrow} X_n$ and $X := \text{Spec}R$.

The exact sequence of sheaves on $X_n$

$$1 \rightarrow \mu_n \rightarrow G_m \overset{\iota_n}{\rightarrow} G_m \rightarrow 1$$

induces a long exact sequence of cohomology

$$0 \rightarrow H^0(X_n, \mu_n) \rightarrow H^0(X_n, G_m) \rightarrow H^0(X_n, G_m) \overset{\delta}{\rightarrow} H^1(X_n, \mu_n) \rightarrow \cdots.$$ Let $(\xi_n)_{n \in \mathbb{N}}$ be a compatible family of $l^n$-th roots of $1$, i.e., $(\xi_{n+1})^l = \xi_n$ for all $n$. The element $\xi_n$ defines an element in $H^0(X_n, \mu_{l^n})$, which we denote also by $\xi_n$. Let $\alpha_n := \xi_n \cup \cdots \cup \xi_n$ (i - 1 times) and let $u_n \in R_n = H^0(X_n; G_m)$. Observe that $\alpha_n \in H^0(X_n, \mu_{l^{(i-1)}})$.

Define

$$s_n^i := N_n(\alpha_n \cup \delta(u_n)),$$

where $N_n : H^1(X_n, \mu_{l^{(i)}}) \rightarrow H^1(X, \mu_{l^{(i)}})$ is the transfer map associated to the etale covering $p_n : X_n \rightarrow X$.

The next lemma one finds in [4].

Lemma 3.1. Let $(u_n)_{n \in \mathbb{N}}$, $u_n \in R_n^\ast$, be a family of $l$-units such that $N_{n+1}(u_{n+1}) = u_n$, where $N_{n+1} : R_n^\ast \rightarrow R_n^\ast$ is the norm. Then $r_{n+1,n}(s_{n+1}) = s_n^i$, where $r_{n+1,n} : H^1(X, \mu_{l^{n+1}}) \rightarrow H^1(X, \mu_{l^n})$ is induced by the projection $\mu_{l^{n+1}} \rightarrow \mu_{l^n}$. Hence the family $(s_n^i)_{n \in \mathbb{N}}$ defines an element

$$s^i((u_n)_{n \in \mathbb{N}}) \in H^1(X, Z_l(i)).$$

The element $s^i = s^i((u_n)_{n \in \mathbb{N}}) \in H^1(X, Z_l(i))$ is called a Soulé class. Further we shall relate Soulé classes to $\ell$-adic polylogarithms.

The covering $p_n : X_n \rightarrow X$ is Galois. It follows from [3] Lemma 6 that we have a commutative diagram

$$\begin{array}{ccc}
(R_n^\ast / (R_n^\ast)^\mu)^{(i-1)} \otimes (\mu_{l^n})^\otimes & \xrightarrow{\delta \otimes \text{id}} & H^1(X_n; \mu_{l^n}^\otimes) \\
N_{\Gamma_n} & & \downarrow \text{Id} \\
(R_n^\ast / (R_n^\ast)^\mu)^{(i-1)} \otimes (\mu_{l^n})^\otimes & \xrightarrow{\delta \otimes \text{id}} & H^1(X_n; \mu_{l^n}^\otimes),
\end{array}$$

where $N_{\Gamma_n}(c) = \prod_{\sigma \in \Gamma_n} \sigma(c)$.

Observe that

$$N_{\Gamma_n}(u_n \otimes \xi_{l^n}^\otimes((i-1))) = \prod_{\sigma \in \Gamma_n} (\sigma(u_n)) \otimes (\xi_{l^n}^\sigma)^\otimes((i-1)) =$$
where $\chi(\sigma)$ is the unique positive integer such that $0 \leq \chi(\sigma) < |\Gamma_n|$ and $\chi(\sigma) \equiv \chi(\sigma) \mod |\Gamma_n|$. Hence we have proved the following result.

**Lemma 3.2.** The element

$$p_n^*(\alpha_i^j) \in H^1(X_n; \mu_{\alpha_i^j}^n) \approx Hom(G_n; \mu_{\alpha_i^j}^n) \otimes \mu_{\alpha_i^j}^{n(i-1)}$$

is given by the Kummer character associated to $\prod_{\sigma \in \Gamma_n} (\sigma(u_n))^{\chi(\sigma)}-1$.

In the sequel we shall restrict our attention to $K = \mathbb{Q}(\mu_r)$. We shall relate Soulé classes associated to families of $l$-units consider in $[5]$ to $l$-adic polylogarithms evaluated at $r$-th roots of 1. In 3.2 we consider the case when $r$ is prime to $l$ and in 3.3 we consider the general case.

3.2. Let $r$ be a positive integer prime to $l$ and let $K = \mathbb{Q}(\mu_r)$. For each integer $q$ such that $0 \leq q < r$ we chose a compatible family $(\xi_{\sigma}^q)_{\sigma \in \Gamma_n}$ of $l^n$-th roots of $\xi_{\sigma}^q$ such that each $\xi_{\sigma}^q \in \mu_r$. Then the family of $l$-units $u_n(q) := (1 - \xi_{\sigma}^q \cdot \xi_{\tau}^q (\sigma, \tau) \in \mathbb{N}$ is compatible i.e. $N_{n+1,n}(1 - \xi_{\sigma}^q \cdot \xi_{\tau}^q (\sigma, \tau) \in (1 - \xi_{\sigma}^q \cdot \xi_{\tau}^q (\sigma, \tau) \in \mathbb{N}$ for all $n \in \mathbb{N}$. It follows from Lemma 3.2 that the corresponding Soulé class $s_n((1 - \xi_{\sigma}^q \cdot \xi_{\tau}^q (\sigma, \tau) \in \mathbb{N}$ taken modulo $l^n$ and restricted to $Gal(M/\mathbb{Q}(\mu_r))$ is given by the Kummer character associated to $\prod_{\sigma \in \Gamma_n} (\sigma(u_n))^{\chi(\sigma)}-1$, where $M$ is a maximal Galois extension of $\mathbb{Q}(\mu_r)$ unramified outside $l$.

Hence we have proved the following result.

**Lemma 3.3.** We have

$$s_n((1 - \xi_{\sigma}^q \cdot \xi_{\tau}^q (\sigma, \tau) \in \mathbb{N})(\sigma) = \int_{\mathbb{Z}_l^q} x^{m-1} dx(\xi_{\sigma}^q (\sigma))$$

for any $\sigma \in Gal(M/\mathbb{Q}(\mu_r))$.

In the next proposition we relate Soulé classes to $l$-adic polylogarithms evaluated at $r$-th roots of 1.

**Proposition 3.4.** (see also $[6]$) We have

$$s_n((1 - \xi_{\sigma}^q \cdot \xi_{\tau}^q (\sigma, \tau) \in \mathbb{N}) = \ell_m(\xi_{\sigma}^q) - l^{m-1} \ell_m(\xi_{\tau}^{q-1}).$$

**Proof.** Observe that

$$\sum_{i=0}^{l^n-1} \kappa_i (\xi_{\sigma}^q) = \sum_{i=0}^{l^n-1} \kappa_i (\xi_{\tau}^q) + \sum_{i=0}^{l^n-1} \kappa_i (\xi_{\sigma}^q)$$

and

$$\kappa_i (\xi_{\sigma}^q) = \kappa_{n-1} (\xi_{\tau}^{q-1}).$$

It follows from 3.1. and 3.2 and from section 1 that

$$\int_{\mathbb{Z}_l^q} x^{m-1} dx(\xi_{\sigma}^q (\sigma)) = \int_{\mathbb{Z}_l^q} x^{m-1} dx(\xi_{\sigma}^q (\sigma)) + l^{m-1} \int_{\mathbb{Z}_l^q} x^{m-1} dx(\xi_{\tau}^{q-1})(\sigma).$$

Hence by the definition of $l$-adic polylogarithms we have

$$\ell_m(\xi_{\sigma}^q) = \int_{\mathbb{Z}_l^q} x^{m-1} dx(\xi_{\sigma}^q (\sigma)) + l^{m-1} \ell_m(\xi_{\tau}^{q-1})(\sigma).$$

The proposition now follows from the equality (3.3) and Lemma 3.3. □

Remark. The special case of this identity appears in the paper of Ihara (see $[1]$ pages 104 and 105).
Corollary 3.5. Let $e$ be the order of $l$ in $(\mathbb{Z}/r)^\ast$. Then
\[
(1 - l^{e(m-1)})\ell_m(\xi^n) = \sum_{i=0}^{e-1} p^{i(m-1)} s^m((1 - \xi^{\frac{b+i}{r}} \cdot \xi^n)_{n \in \mathbb{N}})
\]
on $\text{Gal}(\mathcal{M}/\mathbb{Q}(\mu_{r-1}))$.

3.3. Let $r$ be an integer prime to $l$ and let $\nu$ be a positive integer. Let $K = \mathbb{Q}(\mu_{\nu}, r)$. Then $K_{n+\nu} = \mathbb{Q}(\mu_{n+\nu}, r)$ and $\Gamma_{n+\nu} = \mathbb{Z}/\mathbb{Z}$. Let $\alpha \in \Gamma_{n+\nu} = \mathbb{Z}/\mathbb{Z}$. Then $\alpha(\xi_{n+\nu}) = \xi_{1+\nu}^\alpha$. Let $p$ be an integer prime to $l$ and let $a$ be an integer such that $0 \leq a < r$. The family of $l$-units $(1 - \xi^\frac{b+a}{r} \cdot \xi^n)_{n \geq \nu}$ is compatible, i.e., $N_{n+1,n}(1 - \xi^\frac{b+a}{r} \cdot \xi^n_{1+\nu}) = (1 - \xi^\frac{b+a}{r} \cdot \xi^n_{1+\nu})$ for all $n \geq \nu$. We have
\[
N_{\Gamma_{n+\nu}}((1 - \xi^\frac{b+a}{r} \cdot \xi^n_{1+\nu})\otimes \xi^{\otimes(m-1)}_{n+\nu}) = \prod_{\alpha \in \mathbb{Z}/\mathbb{Z}^\nu} (1 - \xi^\frac{b+a}{r} \cdot \xi^{(1+\nu)\alpha}_n) \otimes (\xi^{(1+\nu)\alpha}_n) \otimes (\xi^{\otimes(m-1)}_{n+\nu})
\]
\[
= \prod_{\alpha=0}^{l^n-1} \left(1 - \xi^\frac{b+a}{r} \cdot \xi^{(1+\nu)\alpha}_n\right) \otimes (\xi^{\otimes(m-1)}_{n+\nu})
\]
\[
= \prod_{k=0}^{m-1} \prod_{\alpha=0}^{l^n-1} \left(1 - \xi^{\frac{b+a}{r} \cdot \xi^\alpha_n \cdot \xi^n_{1+\nu}}\right) \otimes (\xi^{\otimes(m-1)}_{n+\nu})
\]
\[
= \prod_{k=0}^{m-1} \prod_{\alpha=0}^{l^n-1} \left(1 - \xi^{\frac{b+a}{r} \cdot \xi^\alpha_n \cdot \xi^n_{1+\nu}}\right) \otimes (\xi^{\otimes(m-1)}_{n+\nu})
\]

The Kummer character associated to the last term is
\[
\sum_{k=0}^{m-1} \left(\begin{array}{c}
m - 1 \\ k
\end{array}\right) P_k \cdot q^{-k} \sum_{j=0}^{l^n-1} j_k K_j \cdot (\xi^{\frac{b+a}{r} \cdot \xi^n_{1+\nu}})
\]

Hence we get the following result.

Proposition 3.6. We have
\[
s^m((1 - \xi^\frac{b+a}{r} \cdot \xi^n_{1+\nu})_{n \geq \nu}) = \nu^{(m-1)} q^{-\nu} \ell_m(\xi^{a} \cdot \xi^n) + \sum_{k=0}^{m-2} \left(\begin{array}{c}
m - 1 \\ k
\end{array}\right) l^{(m-1)} q^{-k} \ell_{k+1}(\xi^{a} \cdot \xi^n)
\]
on $\text{Gal}(\mathcal{M}/\mathbb{Q}(\mu_{r-1}))$, where $\mathcal{M}$ is a maximal Galois extension of $\mathbb{Q}(\mu_{r-1})$ unramified outside $l$.

In the next resultat, suggested by the referee, we express $l$-adic polylogarithms evaluated at roots of 1 by Soulé classes.

Corollary 3.7. We have
\[
l^{\nu(1)} q^{-\nu} \ell_{m}(\xi^{a} \cdot \xi^n) = \sum_{i=0}^{m-1} (-1)^i \left(\begin{array}{c}
m - 1 \\ i
\end{array}\right) s^{m-i}((1 - \xi^\frac{b+a}{r} \cdot \xi^n_{1+\nu})_{n \geq \nu}).
\]

Proof. The corollary follows from the binomial formulas $(b+1)^n = \sum_{k=0}^{n} \binom{n}{k} b^k$ and $b^n = ((b+1) + (-1))^n = \sum_{k=0}^{n} \binom{n}{k} (a)^{i} (b+1)^{n-i}$. □

The next proposition generalizes Corollary 8.4.2 from [6].

Theorem 3.8. The functions $\ell_{m}(\xi^{a} \cdot \xi^n)$ for $0 < a \leq \frac{l-1}{r}$, $(a,r) = 1$ and $0 < q \leq \frac{l}{r}$, $(q,l) = 1$ are linearly independent over $\mathbb{Z}_l$. 

Proof. It is sufficient to show that the Soulé classes \( x^{\alpha}(1 - \xi_{r}^{\frac{m-\alpha}{2}} \cdot \xi_{r}^{\frac{q}{2}}) \) for \( 0 < a \leq \frac{m-1}{2}, (a, r) = 1 \) and \( 0 < q \leq \frac{r}{2}, (q, l) = 1 \) are linearly independent over \( \mathbb{Z} \).

This follows from the fact that the \( l \)-units \( 1 - \xi_{r}^{\alpha} \cdot \xi_{l}^{\frac{q}{2}} \) for \( 0 < a \leq \frac{m-1}{2}, (a, r) = 1 \) and \( 0 < q \leq \frac{r}{2}, (q, l) = 1 \) are linearly independent in the group of cyclotomic \( l \)-units of \( \mathbb{Q}(\mu_{r^l}) \) tensored by \( \mu_{l}^{\otimes (m-1)} \).

\[ \square \]

References


