THE WIRTHMÜLLER ISOMORPHISM REVISITED

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Abstract. We show how the formal Wirthmüller isomorphism theorem proven in [2] simplifies the proof of the Wirthmüller isomorphism in equivariant stable homotopy theory. Other examples from equivariant stable homotopy theory show that the hypotheses of the formal Wirthmüller and formal Grothendieck isomorphism theorems in [2] cannot be weakened.

Contents

We illustrate the force of the formal Wirthmüller isomorphism theorem of [2] by giving a worked example of an interesting theorem whose proof it simplifies, namely the Wirthmüller isomorphism theorem in equivariant stable homotopy theory. It relates categories of G-spectra and H-spectra for $H \subset G$. We also say just a little about the analogous Adams isomorphism that relates categories of G-spectra and J-spectra for a quotient group J of G. That context gives an interesting situation where the formal hypotheses of the formal Wirthmüller isomorphism theorem hold but the conclusion fails, showing that the more substantive hypothesis is essential.

In their general form, the Wirthmüller and Adams isomorphisms are due to Gaunce Lewis and myself [6]. It is a pleasure to thank Lewis for recent e-mails, ongoing discussions, and his longstanding quest for simplifications and generalizations of these theorems. The analogy between these isomorphisms in topology and Verdier duality was first explored by Po Hu [4], who carried out an idea of Lewis that these isomorphisms could be obtained using parametrized equivariant spectra. She obtained a substantial generalization of the Wirthmüller isomorphism, but at the price of greatly increased complexity. Exactly as in our proof here, the theory of [2] allows considerable simplification of her work, and that is part of our motivation. The theory of [2] should also simplify the proof of the Wirthmüller isomorphism in $\mathbf{A}^1$ stable homotopy theory that Hu proved in [3]. Still another example recently studied by Hu deals with change of universe. We show that it gives an interesting naturally occurring situation in which all but one of the hypotheses of the formal Grothendieck isomorphism of [2] hold, but the conclusion fails because the relevant left adjoint fails to preserve compact objects.

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1. The Wirthmüller isomorphism

Let H be a (closed) subgroup of a compact Lie group G and let $f : H \longrightarrow G$ be the inclusion. Let L be the tangent H-representation at the coset $eH \in G/H$. Thus, if G is finite or, more generally, if H has finite index in G, then $L = 0$.

Let $\mathscr{D}$ and $\mathscr{C}$ be the stable homotopy categories of G-spectra and of H-spectra, as constructed in [6] or, in more modern form, [7]. The category $\mathscr{D}$ depends on a choice of a "G-universe" $\mathscr{V}$ on which to index G-spectra. We may think of $\mathscr{V}$ as a chosen collection $\{V\}$ of G-representations V that contains the trivial representation and is closed under direct sums. The most interesting example is the complete G-universe obtained by allowing all representations of G. We think of representations as finite dimensional G-inner product spaces and let S^V denote the one-point compactification of V. The point of the choice of G-universe is that, in the construction of $\mathscr{D}$, we force suspension by some representations V to be equivalences $\mathscr{D} \longrightarrow \mathscr{D}$, and we must choose which representations to invert in this sense.

We insist that G/H embed in a representation V in our G-universe $\mathscr{V}$, which is otherwise unrestricted. We index H-spectra on the H-universe $f^*\mathscr{V} = \{f^*V\}$, where f^*V denotes the G-representation V viewed as an H-representation by pullback along f. Similarly, a G-spectrum Y gives an H-spectrum f^*Y by pullback along f. The functor f^* has a left adjoint $f_!$ and a right adjoint f_*. The former is usually written as either $G_+ \wedge_H X$ or $G \ltimes_H X$, and the latter is usually written as either $F_H(G_+, X)$ or $F_H[G, X)$. The Wirthmüller isomorphism reads as follows.

Theorem 1.1 (Wirthmüller isomorphism)**.** *There is a natural isomorphism*

$$\omega : f_*X \longrightarrow f_\sharp X, \quad \textit{where} \quad f_\sharp X = f_!(X \wedge S^{-L}).$$

That is, for an H-spectrum X, $F_H(G_+, X) \cong G_+ \wedge_H (\Sigma^{-L}X)$.

Here the suspension H-spectrum of S^L is invertible with inverse S^{-L}, allowing the definition $\Sigma^{-L}X = X \wedge S^{-L}$. Indeed, an embedding of G/H in $V \in \mathscr{V}$ induces an inclusion $L \subset V$ of H-represenions with orthogonal complement W. Since $S^L \wedge S^W \cong S^{f^*V}$ is invertible in $\mathscr{C}$, S^L is also invertible in $\mathscr{C}$.

The unit objects in $\mathscr{C}$ and $\mathscr{D}$ are the sphere spectra S_H and S_G. Both $\mathscr{C}$ and $\mathscr{D}$ are closed symmetric monoidal categories under their smash product and function spectrum functors $\wedge$ and F. It is immediate from the definitions that f^* is strong symmetric monoidal and commutes with F, as required in the Wirthmüller context discussed in [2, §§2, 4].

Remark 1.2. In this context, the projection formula would assert that

$$Y \wedge F_H(G_+, X) \cong F_H(G_+, f^*Y \wedge X),$$

which is false on the spectrum level but which holds on the stable category level as a consequence of the Wirthmüller isomorphism. Note that, on the spectrum level, $F_H(G_+, X)^G \cong X^H$, by a comparison of adjunctions.

The isomorphism [2, 4.3] required in the Wirthmüller context can be written

$$D(G_+ \wedge_H S_H) \cong G_+ \wedge_H S^{-L}. \tag{1.3}$$

It is a special case of equivariant Atiyah duality for smooth G-manifolds, which is proven by standard space level techniques (e.g. [6, III§5]) and is independent of the

Wirthmüller isomorphism. By the tubular neighborhood theorem, we can extend an embedding $i : G/H \longrightarrow V$ to an embedding

$$\tilde{i} : G \times_H W \longrightarrow V \tag{1.4}$$

of the normal bundle $G \times_H W$. Atiyah duality asserts that the G-space G/H_+ is Spanier-Whitehead V-dual to the Thom complex $G_+ \wedge_H S^W$ of the normal bundle of the embedding. Desuspending by S^V in $\mathscr{D}$ gives the required isomorphism (1.3).

The category $\mathscr{D}$ is triangulated, with distinguished triangles isomorphic to canonical cofiber sequences of G-spectra. The triangulation is compatible with the closed symmetric monoidal structure in the sense discussed in [8]. Moreover, $\mathscr{D}$ is compactly generated. Writing S^n for the n-sphere G-spectrum, we can choose the generators to be the G-spectra $G/J_+ \wedge S^n$, where J ranges over the (closed) subgroups of G and n runs over the integers. The same statements apply to $\mathscr{C}$, and the functor f^* is exact since it commutes with cofiber sequences on the level of spectra. Moreover, f^* takes generators to compact objects. Indeed, this depends only on the fact that the H-spaces G/J are compact and of the homotopy types of H-CW complexes, although it is easier to verify using the stronger fact that the G/J can be decomposed as finite H-CW complexes.

Remark 1.5. Observe that the generators H/K_+ of $\mathscr{C}$ and the generators G/J_+ of $\mathscr{D}$, other than G/H_+, need not be dualizable if the universe is incomplete. In fact, Lewis [5, 7.1] has proven that G/J_+ is dualizable if and only if G/J embeds in a representation in $\mathscr{V}$. Conceptually, it is compactness rather than dualizability of the generators that is relevant.

Digressively, there is an interesting conceptual point to be made about the choice of generators. There are (at least) three different, but Quillen equivalent, model categories of G-spectra. We can take "G-spectra" to mean G-spectra as originally defined [6], S_G-modules as defined in [1], or orthogonal G-spectra as defined in [7], where the cited Quillen equivalences are proven. These three categories are also compactly generated in the model theoretic sense. In the first two cases, the generators in the model theoretic sense can be taken to be the generators in the triangulated category sense that we have just specified. As explained in [7], we can alternatively take all $G/J_+ \wedge S^{V-V'}$ for $V, V' \in \mathscr{V}$ as generators in the model theoretic sense. However, in the model category of orthogonal G-spectra, we not only can but must take this larger collection as generators in the model theoretic sense. Nevertheless, the smaller collection suffices to generate the associated triangulated homotopy category, since that is triangulated equivalent to the homotopy category obtained from the other two model categories.

By the formal Wirthmüller isomorphism theorem [2, 6.1], to prove Theorem 1.1, it remains to prove that the map $\omega : f_*X \longrightarrow f_\sharp X$ specified in [2, 4.7] is an isomorphism when X is a generating object $H/K_+ \wedge S^n$ of $\mathscr{C}$. Since it is obvious that ω commutes with suspension and desuspension, we need only consider the case $n = 0$, where the generators in question are the suspension H-spectra $\Sigma^\infty H/K_+$.

Here is the punchline. Suppose that G is finite or, more generally, that H has finite index in G. Then, as an H-space, H/K_+ is a retract of the G-space G/K_+. The retraction sends cosets of G/K_+ not in the image of H/K_+ to the disjoint basepoint of H/K_+. By [2, 4.13], it follows formally that ω is an isomorphism when $X = \Sigma^\infty H/K_+$. This simple argument already completes the proof of Theorem 1.1 in this case.

To prove Theorem 1.1 in general, we apply [2, 4.14], which allows us to work one generating object at a time. This reduces all of our work to consideration of suspension spectra and thus of spaces. The argument is essentially the same as part of the argument in [6, III§§5, 6], but we shall run through the space-level details in §4 in order to have a readable and self-contained account.

2. The Adams isomorphism

Let N be a normal subgroup of a compact Lie group G and let $f : G \longrightarrow J$, $J = G/N$, be the quotient homomorphism. Fix a G-universe $\mathscr{V} = \{V\}$ and index J-spectra on the N-fixed J-universe $\mathscr{V}^N = \{V^N\}$. Regarding J-representations as G-representations via f, we obtain a second G-universe $f^*\mathscr{V}^N = \{f^*V^N\}$, and we insist that $f^*\mathscr{V}^N$ be contained in the original G-universe $\mathscr{V}$. Let $\mathscr{D}$ be the stable homotopy category of J-spectra indexed on $\mathscr{V}^N$ and let $\mathscr{C}$ be the stable homotopy category of G-spectra indexed on $f^*\mathscr{V}^N$.

Regarding J-spectra as G-spectra via f, we obtain a functor $f^* : \mathscr{D} \longrightarrow \mathscr{C}$. Its left adjoint $f_!$ is just the orbit spectrum functor that sends a G-spectrum X indexed on $f^*\mathscr{V}^N$ to X/N. Its right adjoint f_* is just the fixed point spectrum functor that sends X to X^N. The functor f^* is strong symmetric monoidal, the isomorphism [2, 2.6] holds in the form $(f^*Y \wedge X)/N \cong Y \wedge (X/N)$, and f^* takes generating objects to compact objects. Since G-spectra in $\mathscr{C}$ are indexed on an N-trivial universe, $S_G/N \cong S_J$ and [2, 4.3] holds in the trivial form $D(S_J) \cong S_G/N$. Thus all of the formal hypotheses of the formal Wirthmüller isomorphism theorem, [2, 6.1], are satisfied. However, the conclusion fails, because ω is an isomorphism on some but not all generators. Let $\mathscr{C}/N$ be the thick subcategory of $\mathscr{C}$ generated by all $\Sigma^n\Sigma^\infty_G G/H_+$ such that $N \subset H$.

Proposition 2.1. *For $X \in \mathscr{C}/N$, the natural map*

$$\omega : X^N \longrightarrow X/N$$

is an isomorphism of J-spectra.

Proof. The map $\tau : T \longrightarrow f_!C$ of [2, §4] is here just the isomorphism $S_J \cong S_G/N$, and the map ω of [2, 4.7] is just the composite of the isomorphism $X^N \cong f^*X^N/N$ and $\varepsilon/N : f^*X^N/N \longrightarrow X/N$. When $X = \Sigma^n\Sigma^\infty_G G/H_+$ with $N \subset H$, the latter map is also an isomorphism, but it is not an isomorphism in general. □

The Adams isomorphism is more subtle. Let $\mathscr{B}$ be the stable homotopy category of G-spectra indexed on $\mathscr{V}$. We have the adjoint pair (i_*, i^*) of change of universe functors $i_* : \mathscr{C} \longrightarrow \mathscr{B}$ and $i^* : \mathscr{B} \longrightarrow \mathscr{C}$ induced by the inclusion $i : f^*\mathscr{V}^N \longrightarrow \mathscr{V}$. We are most interested in the composite adjunction (i_*f^*, f_*i^*). Thinking of the case when $\mathscr{V}$ is a complete G-universe, it is usual to regard the composite $f_*i^* : \mathscr{B} \longrightarrow \mathscr{D}$ as the N-fixed point spectrum functor from G-spectra indexed on $\mathscr{V}$ to J-spectra indexed on $\mathscr{V}^N$.

The conjugation action of G on N gives rise to an action of G on the tangent space $A = A(N; G)$ of N at e. We call A the adjoint representation of G at N. Of course, $A = 0$ if N is finite. The Adams isomorphism reads as follows. Let $\mathscr{C}_f$ be the full subcategory of N-free G-spectra in $\mathscr{C}$.

Theorem 2.2 (Adams isomorphism)**.** *For $X \in \mathscr{C}_f$, there is a natural isomorphism*

$$\omega : f_*i^*i_*X \longrightarrow f_\sharp X, \quad \textit{where} \quad f_\sharp(X) = f_!(X \wedge S^A).$$

That is, for an N-free G-spectrum X indexed on $f^*\mathscr{V}^N$, $(i^*i_*X)^N \cong \Sigma^A X/N$.

It is usual to write this in the equivalent form $(i^*\Sigma^{-A}i_*X)^N \cong X/N$, but the present form is more convenient for applications and more sensible from the categorical point of view. This looks enough like the formal Wirthmüller isomorphism to expect a similar proof. However, I do not have a helpful formal analysis.

3. Change of universe

Let $i\colon \mathscr{V} \longrightarrow \mathscr{U}$ be a map of G-universes, say for definiteness an inclusion. We have the adjoint pair (i_*, i^*) relating the stable homotopy categories $\mathscr{C}$ and $\mathscr{D}$ of G-spectra indexed on $\mathscr{V}$ and G-spectra indexed on $\mathscr{U}$. The left adjoint i_* preserves compact objects, hence the right adjoint i^* preserves coproducts [2, 5.4]. Since $\mathscr{D}$ is compactly generated, the right adjoint i^* has a right adjoint $i_!$ by the triangulated adjoint functor theorem [2, 6.3]. It occurred to Po Hu to study the structure of such functors $i_!$, and the natural question to ask is whether or not the formal Grothendieck isomorphism theorem [2, 6.4] applies. The functors $(i_*, i^*, i_!)$ here play the roles of the functors $(f^*, f_*, f^!)$ there, and the projection formula takes the form of a natural isomorphism

$$Y \wedge i^*X \cong i^*(i_*Y \wedge X).$$

This holds when Y is a suspension G-spectrum because the suspension G-spectrum functors on the two universes satisfy $i_*\Sigma^\infty \cong \Sigma^\infty$ and i^* commutes with smash products. It therefore holds in general [2, 5.6].

There is a natural map

$$\phi : i_*Y \wedge i_!Z \longrightarrow i_!(Y \wedge Z)$$

which is an isomorphism for all dualizable Y [2, 3.9]. We ask whether or not it is an isomorphism for all Y, and the answer is no. Indeed, the necessary hypothesis that $i_!$ preserves coproducts in [2, 6.4] is satisfied if and only if i^* takes generators to compact objects [2, 5.4], and this fails in general. To see this, let $\mathscr{U}$ be a complete G-universe and $\mathscr{V} = \mathscr{U}^G$ be the G-fixed subuniverse $\{U^G\}$, $U \in \mathscr{U}$. The G-fixed point functor from $\mathscr{D}$ to the stable homotopy category of spectra is the composite of i^* and the G-fixed point spectrum functor from $\mathscr{C}$ to spectra. The latter functor preserves compact objects by inspection. For example, it commutes with the suspension spectrum functor and therefore takes suspension spectra of compact G-spaces to suspension spectra of compact spaces. However, for a based G-space Y, the G-spectrum $\Sigma^\infty Y$ indexed on $\mathscr{U}$ has G-fixed point spectrum the wedge over conjugacy classes (H) of the suspension spectra of the spaces $EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H$ [6, V.11.1]. Even when $Y = S^0$, this spectrum is not compact. Therefore $i^*\Sigma^\infty Y$ cannot be a compact G-spectrum indexed on $\mathscr{U}^G$.

4. Completion of the proof of the Wirthmüller isomorphism

We must verify the hypotheses of [2, 4.14] for $X = \Sigma^\infty H/K_+$. This means that, with $f_\sharp X = G_+ \wedge_H (X \wedge S^{-L})$, we must construct a map $\xi : f^*f_\sharp X \longrightarrow X$ such that certain diagrams commute. We need some space level constructions from [6] to do this. The tubular neighborhood (1.4) gives a Pontryagin-Thom G-map $t : S^V \longrightarrow G_+ \wedge_H S^W$. It is V-dual to the counit G-map $\sigma : f_!f^*S^0 = G/H_+ \longrightarrow S^0$ [6, III.5.2]. The following construction, which is [6, II.5.5], specializes to give the

V-dual $u : G_+ \wedge_H S^W \longrightarrow S^V$ to the unit H-map $\zeta : S^0 \longrightarrow f^* f_! S^0 = G/H_+$. We omit f^* from notations, instead stating the equivariance explicitly.

Construction 4.1. Let $H \times H$ act on G by $(h_1, h_2)g = h_1 g h_2^{-1}$ and act on $L \times H$ by $(h_1, h_2)(\lambda, h) = (h_1\lambda, h_1 h h_2^{-1})$. We think of the first factor H as acting from the left, the second as acting from the right. Using the exponential map, construct an embedding $j : L \longrightarrow G$ of L as a slice at e such that

$$j(h\lambda) = hj(\lambda)h^{-1} \quad \text{and} \quad j(-\lambda) = j(\lambda)^{-1}. \tag{4.2}$$

Define $\tilde{j} : L \times H \longrightarrow G$ by $\tilde{j}(\lambda, h) = j(\lambda)h$. Then $\tilde{j}$ is an $(H \times H)$-map that embeds $L \times H$ onto an open neighborhood of e. Collapsing the complement to a point, we obtain an $(H \times H)$-map $u : G_+ \longrightarrow S^L \wedge H_+$. For a based H-space X, we obtain an induced (left) H-map

$$u : G_+ \wedge_H X \longrightarrow (S^L \wedge H_+) \wedge_H X \cong S^L \wedge X.$$

Setting $X = S^W$ and identifying S^V with $S^L \wedge S^W$, we obtain the promised V-dual $u : G_+ \wedge_H S^W \longrightarrow S^V$ of $\zeta : S^0 \longrightarrow G/H_+$.

We only need the following definition and lemmas for the H-spaces $X = H/K_+$, but it is simpler notationally to proceed more generally. We write suspension coordinates on the right, $\Sigma^V Y = Y \wedge S^V$. This is important to remember for control of signs, which are units in Burnside rings. Recall that, for a G-space Y and an H-space Z, the natural isomorphism of G-spaces

$$\bar{\pi} : G_+ \wedge_H (Y \wedge Z) \longrightarrow Y \wedge (G_+ \wedge_H Z)$$

is given by the formulas

$$\bar{\pi}(g \wedge y \wedge z) = gy \wedge g \wedge z \quad \text{and} \quad \bar{\pi}^{-1}(y \wedge g \wedge z) = g \wedge g^{-1}y \wedge z. \tag{4.3}$$

Definition 4.4. For a based H-CW complex X, define an H-map

$$\xi : f^* f_\sharp \Sigma^\infty_H X \longrightarrow \Sigma^\infty_H X$$

as follows. Observe that we have natural isomorphisms of H-spectra

$$\begin{aligned} \Sigma^V f^* f_\sharp \Sigma^\infty_H X &\cong S^V \wedge (G_+ \wedge_H (X \wedge S^{-L})) \\ &\xrightarrow{\bar{\pi}^{-1}} G_+ \wedge_H (S^V \wedge X \wedge S^{-L}) \cong \Sigma^\infty_H (G_+ \wedge_H (X \wedge S^W)) \end{aligned}$$

and

$$\Sigma^V \Sigma^\infty_H X \cong \Sigma^\infty_H (X \wedge S^V).$$

Let $\Sigma^V \xi$ correspond under these isomorphisms to $\Sigma^\infty_H u$, where

$$u : G_+ \wedge_H (X \wedge S^W) \longrightarrow S^L \wedge X \wedge S^W \cong X \wedge S^V.$$

The following observation is taken from [6, II.5.8].

Lemma 4.5. *For a G-space Y and an H-space Z, the H-map*

$$G_+ \wedge_H (Y \wedge Z) \cong Y \wedge (G_+ \wedge_H Z) \xrightarrow{\mathrm{id} \wedge u} Y \wedge S^L \wedge Z$$

is canonically H-homotopic to the H-map

$$G_+ \wedge_H (Y \wedge Z) \xrightarrow{u} S^L \wedge Y \wedge Z \cong Y \wedge S^L \wedge Z.$$

Therefore, for any H-map $\theta : Y \longrightarrow X$ from a G-space Y to an H-space X, the following diagram is canonically homotopy commutative.

$$\begin{tikzcd}
G_+ \wedge_H (Y \wedge Z) \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{id} \wedge (\theta \wedge \mathrm{id})"'] & Y \wedge (G_+ \wedge_H Z) \arrow[r, "\mathrm{id} \wedge u"] & Y \wedge S^L \wedge Z \arrow[d, "\theta \wedge \mathrm{id}"] \\
G_+ \wedge_H (X \wedge Z) \arrow[r, "u"'] & S^L \wedge X \wedge Z \arrow[r, phantom, "\cong"] & X \wedge S^L \wedge Z
\end{tikzcd}$$

Proof. Both maps send all points of G not in $\tilde{j}(L \times H)$ to the basepoint. The first takes $\tilde{j}(\lambda, h) \wedge y \wedge z$ to $j(\lambda)hy \wedge \lambda \wedge hz$ and the second takes it to $hy \wedge \lambda \wedge hz$. Applying j to a contracting homotopy of L, we obtain an H-homotopy from $j : L \longrightarrow G$ to the constant map at e, giving the required homotopy. The last statement follows since the diagram commutes by naturality if its top arrow is replaced by u. $\square$

This leads to the following naturality statement. The partial naturality diagram [2, 4.16] is the case in which $Y = f_* X$ and θ is the counit of the (f^*, f_*) adjunction.

Lemma 4.6. *Let Y be a G-spectrum and X be a based H-space. For any map $\theta : Y \longrightarrow \Sigma_H^\infty X$ of H-spectra, the following diagram of H-spectra commutes in $\mathscr{C}$.*

$$\begin{tikzcd}
G_+ \wedge_H (Y \wedge S^{-L}) \arrow[r, "\xi"] \arrow[d, "\mathrm{id} \wedge (\theta \wedge \mathrm{id})"'] & Y \arrow[d, "\theta"] \\
G_+ \wedge_H (\Sigma_H^\infty X \wedge S^{-L}) \arrow[r, "\xi"'] & \Sigma_H^\infty X
\end{tikzcd}$$

Proof. The upper map ξ in the diagram is defined formally in [2, 4.9]. It suffices to prove that the diagram commutes after suspension by V. This has the effect of replacing S^{-L} by S^W on the left and suspending by V on the right. Comparing [2, 4.9] with Definition 4.4 and taking $Z = S^W$, the conclusion reduces to application of the spacewise diagram of the previous lemma to the spaces that comprise the given spectra. Technically, this is most easily seen using prespectra or orthogonal spectra rather than actual spectra, but the essential point is just that the homotopy in the previous lemma is sufficiently natural to commute with the structure maps. $\square$

To complete the proof of Theorem 1.1, it suffices to show that the following specialization of the diagram [2, 4.17] commutes.

(4.7)
$$\begin{tikzcd}
\Sigma_H^\infty X \wedge S^{-L} \arrow[r, "\zeta"] \arrow[d, "\zeta"'] & f^* f_!(\Sigma_H^\infty X \wedge S^{-L}) \\
f^* f_!(\Sigma_H^\infty X \wedge S^{-L}) \arrow[r, "f^*\tau"'] & f^* f_!(f^* f_!(\Sigma_H^\infty X \wedge S^{-L}) \wedge S^{-L}) \arrow[u, "f^* f_!(\xi \wedge \mathrm{id})"']
\end{tikzcd}$$

Here $f_!(-) = G_+ \wedge_H (-)$, f^* is the forgetful functor, ζ is the unit of the $(f_!, f^*)$ adjunction, and τ is the map defined in [2, 4.6] with $Y = f_!(\Sigma_H^\infty X \wedge S^{-L})$. We shall see that this reduces to the following space level observations from [6, II§5].

Lemma 4.8. *The following composite is H-homotopic to the identity map.*

$$S^V \xrightarrow{t} G_+ \wedge_H S^W \xrightarrow{u} S^V.$$

Proof. Composing the embeddings $\tilde{i} : G \times_H W \longrightarrow V$ and $\tilde{j} : L \times H \longrightarrow G$, we obtain an embedding $V = L \times W = (L \times H) \times_H W \longrightarrow G \times_H V$. The composite $u \circ t$ is k^{-1} on $k(V)$, and it collapses the complement of $k(V)$ to the basepoint. The embedding k is isotopic to the identity, and application of the Pontryagin-Thom construction to an isotopy gives a homotopy $\mathrm{id} \simeq u \circ t$. $\square$

Lemma 4.9. *For an H-space X, the following diagram is H-homotopy commutative. Here $\sigma : S^L \longrightarrow S^L$ maps λ to $-\lambda$.*

$$\begin{tikzcd}
\Sigma^V X \arrow[r, "\Sigma^V \zeta"] \arrow[dd, "\Sigma^V \zeta"'] & \Sigma^V(G_+ \wedge_H X) \cong S^V \wedge (G_+ \wedge_H X) & G_+ \wedge_H (S^V \wedge X) \arrow[l, "\bar{\pi}^{-1}"'] \\
 & & G_+ \wedge_H \Sigma^W(S^L \wedge X) \arrow[u, "\cong"'] \\
\Sigma^V(G_+ \wedge_H X) \arrow[r, "\mathrm{id} \wedge t"'] & (G_+ \wedge_H X) \wedge (G_+ \wedge_H S^W) \arrow[r, "\bar{\pi}^{-1}"'] & G_+ \wedge_H \Sigma^W(G_+ \wedge_H X) \arrow[u, "\mathrm{id} \wedge \Sigma^W(\sigma \wedge \mathrm{id})u"']
\end{tikzcd}$$

Proof. The composite around the bottom maps all points with V coordinate not in $\tilde{i}(j(L) \times W))$ to the basepoint. It maps the point $x \wedge \tilde{i}(j(\lambda), w)$ to $(j(\lambda, x) \wedge j(\lambda)(\lambda, w)$; the sign map σ enters due to (4.2) and (4.3). The H-contractibility of L implies that an H-homotopic map is obtained if we replace $f(\lambda)$ by the identity element $e \in G$. Thus the composite around the right is H-homotopic to $\zeta \wedge u \circ t$, which is H-homotopic to the identify by Lemma 4.8. $\square$

Proof of Theorem 1.1. We must show that the diagram (4.7) commutes. It suffices to prove this after suspending by V and replacing X by $\Sigma^V X$. This has the effect of replacing the H-spectra S^{-L} that appear in the diagram by the H-spaces S^W. Since the functors appearing in the diagram commute with the respective suspension spectrum functors, this reduces the problem to the space level. Here a slightly finicky diagram chase, which amounts to a check of signs coming from permutations of suspension coordinates, shows that the resulting diagram commutes by Lemma 4.9. One point is that $\sigma \wedge \mathrm{id} : S^L \wedge S^L \longrightarrow S^L \wedge S^L$ is H-homotopic to the transposition via the homotopy given by multiplying by the (2×2)-matrices with rows $(-t, 1-t)$ and $(1-t, t)$ for $t \in I$. Another is that we must apply Lemma 4.9 with X replaced by $X \wedge S^W$, which has the effect of introducing a permutation of S^L past S^W. In more detail, after applying Σ^∞_H and replacing X by $X \wedge S^W$, the bottom composite in the diagram of the previous lemma agrees with

$$\Sigma^V \tau : \Sigma^V(G_+ \wedge_H (\Sigma^V X \wedge S^{-L})) \longrightarrow \Sigma^V(G_+ \wedge_H (G_+ \wedge_H (\Sigma^V X \wedge S^{-L}) \wedge S^{-L})$$

under the evident isomorphisms of its domain and target. The interpretation of the right vertical composite is trickier, because of the specification of

$$\xi : G_+ \wedge_H (\Sigma^V X \wedge S^{-L}) \longrightarrow \Sigma^V X$$

in Definition 4.4. A diagram chase after suspending by V shows that ξ agrees under the evident isomorphism of its source with

$$(\sigma \wedge \mathrm{id}) \circ u : G_+ \wedge_H (X \wedge S^W) \longrightarrow S^L \wedge X \wedge S^W \cong \Sigma^V X.$$

Notice that we have evident isomorphisms

$$\Sigma^\infty_H \Sigma^V(G_+ \wedge_H (X \wedge S^W)) \cong \Sigma^V(G_+ \wedge_H (\Sigma^V X \wedge S^{-L})) \cong \Sigma^\infty_H G_+ \wedge_H (\Sigma^V X \wedge S^W),$$

the first of which internally expands S^W to $S^V \wedge S^{-L}$ and the second of which leaves $\Sigma^V X$ alone but uses $\bar{\pi}$ to bring S^V inside and contracts $S^{-L} \wedge S^V$ to S^W. Their composite enters into the upper right corner of the required diagram chase; the transpositions that appear cancel out others, resulting in a sign free conclusion. With these indications, the rest is routine. □

Remark 4.10. Effectively, this proof uses the space level arguments of [6, II§5], but eliminates the need for the spectrum level arguments of [6, II§6]. We warn the reader that [6, II.5.2] and hence the first diagram of [6, II.6.12] are incorrect. Fortunately, they are also irrelevant.

References

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