ISOMORPHISMS BETWEEN LEFT AND RIGHT ADJOINTS

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Abstract. There are many contexts in algebraic geometry, algebraic topology, and homological algebra where one encounters a functor that has both a left and right adjoint, with the right adjoint being isomorphic to a shift of the left adjoint specified by an appropriate "dualizing object". Typically the left adjoint is well understood while the right adjoint is more mysterious, and the result identifies the right adjoint in familiar terms. We give a categorical discussion of such results. One essential point is to differentiate between the classical framework that arises in algebraic geometry and a deceptively similar, but genuinely different, framework that arises in algebraic topology. Another is to make clear which parts of the proofs of such results are formal. The analysis significantly simplifies the proofs of particular cases, as we illustrate in a sequel discussing applications to equivariant stable homotopy theory.

Contents

We give a categorical discussion of Verdier and Grothendieck isomorphisms on the one hand and formally analogous results whose proofs involve different issues on the other. Our point is to explain and compare the two contexts and to differentiate the formal issues from the substantive issues in each. The philosophy goes back to Grothendieck's "six operations" formalism. We fix our general framework, explain what the naive versions of our theorems say, and describe which parts of their proofs are formal in §§1–4. This discussion does not require triangulated categories. Its hypotheses and conclusions make sense in general closed symmetric monoidal categories, whether or not triangulated. In practice, that means that the arguments apply equally well before or after passage to derived categories.

After giving some preliminary results about triangulated categories in §5, we explain the formal theorems comparing left and right adjoints in §6. Our "formal Grothendieck isomorphism theorem" is an abstraction of results of Amnon Neeman, and our "formal Wirthmüller isomorphism theorem" borrows from his ideas. His

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paper [22] has been influential, and he must be thanked for catching a mistake in a preliminary version by the third author. We thank Gaunce Lewis for discussions of the topological context, and we thank Sasha Beilinson and Madhav Nori for making clear that, contrary to our original expectations, the context encountered in algebraic topology is not part of the classical context familiar to algebraic geometers. We also thank Johann Sigurdsson for corrections and emendations.

1. The starting point: an adjoint pair (f^*, f_*)

We fix closed symmetric monoidal categories $\mathscr{C}$ and $\mathscr{D}$ with respective unit objects S and T. We write $\otimes$ and Hom for the product and *internal* hom functor in either category, and we write X (sometimes also W) and Y (sometimes also Z) generically for objects of $\mathscr{C}$ and objects of $\mathscr{D}$, respectively. We write $\mathscr{C}(W,X)$ and $\mathscr{D}(Y,Z)$ for the categorical hom sets. We let $DX = \operatorname{Hom}(X,S)$ denote the dual of X. We let $\mathrm{ev}\colon \operatorname{Hom}(X,W)\otimes X \longrightarrow W$ denote the evaluation map, that is, the counit of the $(\otimes, \operatorname{Hom})$ adjunction

$$\mathscr{C}(X\otimes X', W) \cong \mathscr{C}(X, \operatorname{Hom}(X', W)).$$

We also fix a strong symmetric monoidal functor $f^*\colon \mathscr{D} \longrightarrow \mathscr{C}$. This means that we are given isomorphisms

$$f^*T \cong S \quad \text{and} \quad f^*(Y\otimes Z) \cong f^*Y \otimes f^*Z, \tag{1.1}$$

the second natural, that commute with the associativity, commutativity, and unit isomorphisms for $\otimes$ in $\mathscr{C}$ and $\mathscr{D}$. We assume throughout that f^* has a right adjoint f_*, and we write

$$\varepsilon\colon f^*f_*X \longrightarrow X \quad \text{and} \quad \eta\colon Y \longrightarrow f_*f^*Y$$

for the counit and unit of the adjunction. This general context is fixed throughout.

The notation (f^*, f_*) meshes with standard notation in algebraic geometry, where one starts with a map $f\colon A \longrightarrow B$ of spaces or schemes and f^* and f_* are pullback and pushforward functors on sheaves. In our generality there need be no underlying map "f" in sight. Some simple illustrative examples are given in §3.

The assumption that f^* is strong symmetric monoidal has several basic implications. To begin with, the adjoints of the isomorphism $f^*T \cong S$ and the map

$$f^*(f_*W \otimes f_*X) \cong f^*f_*W \otimes f^*f_*X \xrightarrow{\varepsilon\otimes\varepsilon} W\otimes X$$

are maps

$$T \longrightarrow f_*S \quad \text{and} \quad f_*W\otimes f_*X \longrightarrow f_*(W\otimes X). \tag{1.2}$$

These are not usually isomorphisms. This means that f_* is *lax* symmetric monoidal.

The adjoint of the map

$$f^*\operatorname{Hom}(Y,Z)\otimes f^*Y \cong f^*(\operatorname{Hom}(Y,Z)\otimes Y) \xrightarrow{f^*(\mathrm{ev})} f^*Z$$

is a natural map

$$\alpha\colon f^*\operatorname{Hom}(Y,Z) \longrightarrow \operatorname{Hom}(f^*Y, f^*Z). \tag{1.3}$$

It may or may not be an isomorphism in general, and we say that f^* is *closed symmetric monoidal* if it is. However, the adjoint of the composite map

$$f^*\operatorname{Hom}(Y, f_*X) \xrightarrow{\alpha} \operatorname{Hom}(f^*Y, f^*f_*X) \xrightarrow{\operatorname{Hom}(\mathrm{id},\varepsilon)} \operatorname{Hom}(f^*Y, X)$$

is a natural isomorphism

$$\mathrm{Hom}(Y, f_*X) \cong f_* \mathrm{Hom}(f^*Y, X). \tag{1.4}$$

In particular, $\mathrm{Hom}(Y, f_*S) \cong f_*Df^*Y$. Indeed, we have the following two chains of isomorphisms of functors.

$$\mathscr{D}(Z, \mathrm{Hom}(Y, f_*X)) \cong \mathscr{D}(Z \otimes Y, f_*X) \cong \mathscr{C}(f^*(Z \otimes Y), X)$$

$$\mathscr{D}(Z, f_* \mathrm{Hom}(f^*Y, X)) \cong \mathscr{C}(f^*Z, \mathrm{Hom}(f^*Y, X)) \cong \mathscr{C}(f^*Z \otimes f^*Y, X)$$

By the Yoneda lemma and a check of maps, these show immediately that the assumed isomorphism of functors in (1.1) is *equivalent* to the claimed isomorphism of functors (1.4). That is, the isomorphism of left adjoints in (1.1) is adjoint to the isomorphism of right adjoints in (1.4). Systematic recognition of such "conjugate" pairs of isomorphisms can substitute for quite a bit of excess verbiage in the literature. We call this a "comparison of adjoints" and henceforward leave the details of such arguments to the reader.

Using the isomorphism (1.4), we obtain the following map β, which is analogous to both α and the map of (1.2). Like the latter, it is not usually an isomorphism.

$$\beta\colon f_* \mathrm{Hom}(X, W) \xrightarrow{f_* \mathrm{Hom}(\varepsilon, \mathrm{id})} f_* \mathrm{Hom}(f^*f_*X, W) \xrightarrow{\cong} \mathrm{Hom}(f_*X, f_*W). \tag{1.5}$$

Using (1.2), we also obtain a natural composite

$$\pi\colon Y \otimes f_*X \xrightarrow{\eta \otimes \mathrm{id}} f_*f^*Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X). \tag{1.6}$$

Like α, it may or may not be an isomorphism in general. When it is, we say that the *projection formula* holds.

As noted by Lipman [1, p. 119], there is already a non-trivial "coherence problem" in this general context, the question of determining which compatibility diagrams relating the given data necessarily commute. An early reference for coherence in closed symmetric monoidal categories is [8], and the volume [14] contains several papers on the subject and many references. In particular, a paper of G. Lewis in [14] gives a partial coherence theorem for closed monoidal functors. The categorical literature of coherence is relevant to the study of "compatibilities" that focuses on base change maps and plays an important role in the literature in algebraic geometry (e.g. [1, 4, 7, 6, 11]). A study of that is beyond the scope of this note. A full categorical coherence theorem is not known and would be highly desirable.

We illustrate by recording a particular commutative coherence diagram, namely

$$\begin{tikzcd} f^*DY \otimes f^*Y \arrow[r, "\cong"] \arrow[d, "\alpha \otimes \mathrm{id}"'] & f^*(DY \otimes Y) \arrow[r, "f^*(ev)"] & f^*T \arrow[d, "\cong"] \\ Df^*Y \otimes f^*Y \arrow[rr, "ev"'] & & S. \end{tikzcd} \tag{1.7}$$

We shall need a consequence of this diagram. There is a natural map

$$\nu\colon DX \otimes W \longrightarrow \mathrm{Hom}(X, W),$$

namely the adjoint of

$$DX \otimes W \otimes X \cong DX \otimes X \otimes W \xrightarrow{ev \otimes \mathrm{id}} S \otimes W \cong W.$$

The commutativity of the diagram (1.7) implies the commutativity of the diagram

$$\begin{tikzcd} f^*DY\otimes f^*Z \arrow[r, "\cong"] \arrow[d, "\alpha\otimes\mathrm{id}"'] & f^*(DY\otimes Z) \arrow[r, "f^*\nu"] & f^*\operatorname{Hom}(Y,Z) \arrow[d, "\alpha"] \\ Df^*Y\otimes f^*Z \arrow[rr, "\nu"'] & & \operatorname{Hom}(f^*Y, f^*Z). \end{tikzcd} \tag{1.8}$$

We assume familiarity with the theory of "dualizable" (alias "strongly dualizable" or "finite") objects; see [18] for a recent exposition. The defining property is that $\nu\colon DX\otimes X \longrightarrow \operatorname{Hom}(X,X)$ is an isomorphism. It follows that ν is an isomorphism if either X or W is dualizable. It also follows that the natural map $\rho : X \longrightarrow DDX$ is an isomorphism, but the converse fails in general. When X' is dualizable, we have the duality adjunction

$$\mathscr{C}(X\otimes X', X'') \cong \mathscr{C}(X, DX'\otimes X''). \tag{1.9}$$

As observed in [16, III.1.9], (1.1) and the definitions imply the following result.

Proposition 1.10. *If $Y\in\mathscr{D}$ is dualizable, then DY, f^*Y, and Df^*Y are dualizable and the map α of (1.3) restricts to an isomorphism*

$$f^*DY \cong Df^*Y. \tag{1.11}$$

This implies that α and π are often isomorphisms for formal reasons.

Proposition 1.12. *If $Y\in\mathscr{D}$ is dualizable, then*

$$\alpha\colon f^*\operatorname{Hom}(Y,Z)\longrightarrow \operatorname{Hom}(f^*Y,f^*Z) \quad \textit{and} \quad \pi\colon Y\otimes f_*X \longrightarrow f_*(f^*Y\otimes X)$$

are isomorphisms for all objects $X\in\mathscr{C}$ and $Z\in\mathscr{D}$. Thus, if all objects of $\mathscr{D}$ are dualizable, then f^* is closed symmetric monoidal and the projection formula holds.

Proof. For the first statement, α coincides with the composite

$$f^*\operatorname{Hom}(Y,Z)\cong f^*(DY\otimes Z)\cong f^*DY\otimes f^*Z\cong Df^*Y\otimes f^*Z\cong \operatorname{Hom}(f^*Y,f^*Z).$$

For the second statement, π induces the isomorphism of represented functors

$$\begin{aligned}\mathscr{D}(Z, Y\otimes f_*X) &\cong \mathscr{D}(Z\otimes DY, f_*X)\cong \mathscr{C}(f^*(Z\otimes DY),X)\cong \mathscr{C}(f^*Z\otimes f^*DY, X)\\ &\cong \mathscr{C}(f^*Z\otimes Df^*Y,X)\cong \mathscr{C}(f^*Z, f^*Y\otimes X)\cong \mathscr{D}(Z, f_*(f^*Y\otimes X)). \quad \square\end{aligned}$$

2. The general context: adjoint pairs (f^*, f_*) and $(f_!, f^!)$

In addition to the adjoint pair (f^*, f_*) of the previous section, we here assume given a second adjoint pair $(f_!, f^!)$ relating $\mathscr{C}$ and $\mathscr{D}$, with $f_!\colon \mathscr{C}\longrightarrow\mathscr{D}$ being the left adjoint. We write

$$\sigma\colon f_!f^!Y\longrightarrow Y \quad\text{and}\quad \zeta\colon X\longrightarrow f^!f_!X$$

for the counit and unit of the second adjunction.

The adjunction $\mathscr{D}(Y, f_*X)\cong\mathscr{C}(f^*Y, X)$ can be recovered from the more general "internal Hom adjunction" $\operatorname{Hom}(Y, f_*X)\cong f_*\operatorname{Hom}(f^*Y,X)$ of (1.4) by applying the functor $\mathscr{D}(T,-)$ and using the assumption that $f^*T\cong S$. Analogously, it is natural to hope that the adjunction $\mathscr{D}(f_!X,Y)\cong\mathscr{C}(X,f^!Y)$ can be recovered by applying the functor $\mathscr{D}(T,-)$ to a similar internal Hom adjunction

$$\operatorname{Hom}(f_!X,Y)\cong f_*\operatorname{Hom}(X,f^!Y).$$

However, unlike (1.4), such an adjunction does not follow formally from our hypotheses. Motivated by different specializations of the general context, we consider two triads of basic natural maps that we might ask for relating our four functors. For the first triad, we might ask for either of the following two duality maps, the first of which is a comparison map for the desired internal Hom adjunction.

$$\gamma\colon f_* \operatorname{Hom}(X, f^! Y) \longrightarrow \operatorname{Hom}(f_! X, Y). \tag{2.1}$$

$$\delta\colon \operatorname{Hom}(f^* Y, f^! Z) \longrightarrow f^! \operatorname{Hom}(Y, Z). \tag{2.2}$$

We might also ask for a *projection formula map*

$$\hat{\pi}\colon Y \otimes f_! X \longrightarrow f_!(f^* Y \otimes X), \tag{2.3}$$

which should be thought of as a generalized analogue of the map π of (1.6). These three maps are not formal consequences of the given adjunctions, but rather must be constructed by hand. However, it suffices to construct any one of them.

Proposition 2.4. *Suppose given any one of the natural maps γ, δ, and $\hat{\pi}$. Then it determines the other two by conjugation. The map δ is an isomorphism for all dualizable Y if and only if its conjugate $\hat{\pi}$ is an isomorphism for all dualizable Y. If any one of the three conjugately related maps is a natural isomorphism, then so are the other two.*

The second triad results from the first simply by changing the direction of the arrows. That is, we can ask for natural maps in the following directions.

$$\bar{\gamma}\colon \operatorname{Hom}(f_! X, Y) \longrightarrow f_* \operatorname{Hom}(X, f^! Y). \tag{2.5}$$

$$\bar{\delta}\colon f^! \operatorname{Hom}(Y, Z) \longrightarrow \operatorname{Hom}(f^* Y, f^! Z). \tag{2.6}$$

$$\bar{\pi}\colon f_!(f^* Y \otimes X) \longrightarrow Y \otimes f_! X. \tag{2.7}$$

Here $\bar{\delta}$ is to be viewed as a generalized analogue of the map α of (1.3).

Proposition 2.8. *Suppose given any one of the natural maps $\bar{\gamma}$, $\bar{\delta}$, and $\bar{\pi}$. Then it determines the other two by conjugation. The map $\bar{\delta}$ is an isomorphism for all dualizable Y if and only if its conjugate $\bar{\pi}$ is an isomorphism for all dualizable Y. If any one of the three conjugately related maps is a natural isomorphism, then so are the other two.*

Of course, when the three maps are isomorphisms, the triads are inverse to each other and there is no real difference. However, there are two very different interesting specializations: we might have $f_! = f_*$, or we might have $f^! = f^*$. The first occurs frequently in algebraic geometry, and is familiar. The second occurs in algebraic topology and elsewhere, but seems less familiar. With the first specialization, the first triad of maps arises formally since we can take $\hat{\pi}$ to be the map π of (1.6). With the second specialization, the second triad arises formally since we can take $\bar{\delta}$ to be the map α of (1.3). Recall the isomorphism (1.4), the map β of (1.5), and Proposition 1.12.

Proposition 2.9. *Suppose $f_! = f_*$. Taking $\hat{\pi}$ to be the projection map π of (1.6), the conjugate map γ is the composite*

$$f_* \operatorname{Hom}(X, f^! Y) \xrightarrow{\beta} \operatorname{Hom}(f_* X, f_* f^! Y) \xrightarrow{\operatorname{Hom}(\mathrm{id}, \sigma)} \operatorname{Hom}(f_* X, Y)$$

and the conjugate map δ is the adjoint of the map

$$f_*\operatorname{Hom}(f^*Y, f^!Z) \cong \operatorname{Hom}(Y, f_*f^!Z) \xrightarrow{\operatorname{Hom}(\mathrm{id},\sigma)} \operatorname{Hom}(Y, Z).$$

Moreover, π and δ are isomorphisms if Y is dualizable.

When $f^! = f^*$, passage to adjoints from $S \cong f^*T$ and the natural map

$$W \otimes X \xrightarrow{\zeta\otimes\zeta} f^*f_!W \otimes f^*f_!X \cong f^*(f_!W \otimes f_!X)$$

gives maps, not usually isomorphisms,

$$f_!S \longrightarrow T \quad \text{and} \quad f_!(W \otimes X) \longrightarrow f_!W \otimes f_!X. \tag{2.10}$$

This means that $f_!$ is an *op-lax* symmetric monoidal functor.

Proposition 2.11. *Suppose $f^! = f^*$. Taking $\bar{\delta}$ to be the map α of (1.3), the conjugate map $\bar{\pi}$ is the composite*

$$f_!(f^*Y \otimes X) \longrightarrow f_!f^*Y \otimes f_!X \xrightarrow{\sigma\otimes\mathrm{id}} Y \otimes f_!X$$

and the conjugate map $\bar{\gamma}$ is the adjoint of the map

$$f^*\operatorname{Hom}(f_!X, Y) \xrightarrow{\alpha} \operatorname{Hom}(f^*f_!X, f^*Y) \xrightarrow{\operatorname{Hom}(\zeta,\mathrm{id})} \operatorname{Hom}(X, f^*Y).$$

Moreover α and $\bar{\pi}$ are isomorphisms if Y is dualizable.

Definition 2.12. We introduce names for the different contexts in sight.
(i) *The Verdier-Grothendieck context:* There is a natural isomorphism $\hat{\pi}$ as in (2.3) (projection formula); taking $\bar{\pi} = \hat{\pi}^{-1}$, there are conjugately determined natural isomorphisms $\gamma = \bar{\gamma}^{-1}$, and $\delta = \bar{\delta}^{-1}$.
(ii) *The Grothendieck context:* $f_! = f_*$ and the projection formula holds.
(iii) *The Wirthmüller context:* $f^! = f^*$ and f^* is closed symmetric monoidal.

Thus, in the Grothendieck context, the strong symmetric monoidal functor f^* is the left adjoint of a left adjoint. In the Wirthmüller context, it is a left and a right adjoint.

The Verdier-Grothendieck context encapsulates the properties that hold for suitable derived categories $\mathscr{C}$ and $\mathscr{D}$ of sheaves over locally compact spaces A and B and maps $f : A \longrightarrow B$; see [2, 13, 25]. Here $f_!$ is given by pushforward with compact supports. The same abstract context applies to suitable derived categories $\mathscr{C}$ and $\mathscr{D}$ of complexes of $\mathscr{O}_A$-modules and of $\mathscr{O}_B$-modules for schemes A and B and maps $f : A \longrightarrow B$. In either context, we have $f_! = f_*$ when the map f is proper. For the scheme theoretic context, see [5, 6, 11] and, for more recent reworkings and generalizations, [1, 4, 15, 22]. There is a highly non-trivial categorical, more precisely 2-categorical, coherence problem concerning composites of base change functors in the Verdier-Grothendieck context. A start on this has been made by Voevodsky [7]. Since his discussion focuses on base change relating quadruples $(f^*, f_*, f_!, f^!)$, ignoring $\otimes$ and Hom, it is essentially disjoint from our discussion. The relevant coherence problem simplifies greatly in either the Grothendieck or the Wirthmüller context, due to the canonicity of the maps in Propositions 2.9 and 2.11.

We repeat that our categorical results deduce formal conclusions from formal hypotheses and therefore work equally well before or after passage to derived categories. Much of the work in passing from categories of sheaves to derived categories can be viewed as the verification that formal properties in the category of sheaves

carry over to the same formal properties in derived categories, although other properties only hold after passage to derived categories. A paper by Lipman in [1] takes a similarly categorical point of view.

While the proofs of Propositions 2.4 and 2.9 are formal, in the applications to algebraic geometry they require use of *unbounded* derived categories, since otherwise we would not have closed symmetric monoidal categories to begin with. These were not available until Spaltenstein's paper [24], and he noticed one of our formal implications [24, §6]. Unfortunately, as he makes clear, in the classical sheaf context his methods fail to give the $(f_!, f^!)$ adjunction for all maps f between locally compact spaces. It seems possible that a model theoretic approach to unbounded derived categories would allow one to resolve this problem. In any case, a complete reworking of the theory in model theoretical terms would be of considerable value.

In the algebraic geometry setting, smooth maps lead to a context close to the Wirthmüller context, but that is not our motivation. In that context, we think of f^* as a forgetful functor which does not alter underlying structure, $f_!$ as a kind of extension of scalars functor, and f_* as a kind of "coextension of scalars" functor.

For example, let $f : H \longrightarrow G$ be an inclusion of a subgroup in a group G and let $\mathscr{C}$ and $\mathscr{D}$ be the categories of H-objects and G-objects in some Cartesian closed category, such as topological spaces. Let $f^* : \mathscr{D} \longrightarrow \mathscr{C}$ be the evident forgetful functor. Certainly

$$f^*(Y \times Z) \cong f^*Y \times f^*Z.$$

The left and right adjoints of f^* send an H-object X to $G \times_H X$ and to $\mathrm{Map}_H(G, X)$. Clearly $G \times_H (X \times X')$ is not isomorphic to $(G \times_H X) \times (G \times_H X')$. Our motivating example is a spectrum level analogue of this for which there is a Wirthmüller isomorphism theorem [16, 27]. Our formal Wirthmüller isomorphism theorem below substantially simplifies its proof [20]. One can hope for such a result in any context where group actions and triangulated categories mix.

For another example, let $f : A \longrightarrow B$ be an inclusion of cocommutative Hopf algebras over a field k and let $\mathscr{C}$ and $\mathscr{D}$ be the categories of A-modules and of B-modules. These are closed symmetric monoidal categories under the functors $\otimes_k$ and Hom_k. Indeed, using the coproduct on A, we see that if M and N are A-modules, then so are $M \otimes_k N$ and $\mathrm{Hom}_k(M, N)$. The commutativity of $\otimes_k$ requires the cocommutativity of A. The unit object in both $\mathscr{C}$ and $\mathscr{D}$ is k. Again, if $f^* : \mathscr{D} \longrightarrow \mathscr{C}$ is the evident forgetful functor, then

$$f^*(Y \otimes_k Z) = f^*Y \otimes_k f^*Z.$$

The left and right adjoints of f^* send an A-module X to $B \otimes_A X$ and to $\mathrm{Hom}_A(B, X)$, and again $B \otimes_A (X \otimes_k X')$ is not isomorphic to $(B \otimes_A X) \otimes_k (B \otimes_A X')$. This example deserves investigation on the level of derived categories.

3. Isomorphisms in the Verdier–Grothendieck context

We place ourselves in the Verdier–Grothendieck context in this section.

Definition 3.1. For an object $W \in \mathscr{C}$, define $D_W X = \mathrm{Hom}(X, W)$, the *$W$-twisted dual of X*. Of course, if X or W is dualizable, then $D_W X \cong DX \otimes W$. Let $\rho_W : X \longrightarrow D_W D_W X$ be the adjoint of the evaluation map $D_W X \otimes X \longrightarrow W$. We say that X is *W-reflexive* if ρ_W is an isomorphism.

Replacing Y by Z in (2.1) and letting $W = f^!Z$, the isomorphisms γ and δ take the following form:

$$f_*D_WX \cong D_Zf_!X \quad\text{and}\quad D_Wf^*Y \cong f^!D_ZY. \tag{3.2}$$

This change of notation and comparison with the classical context of algebraic geometry explains why we think of γ and δ as duality maps. If $f_!X$ is Z-reflexive, the first isomorphism implies that

$$f_!X \cong D_Zf_*D_WX. \tag{3.3}$$

If Y is isomorphic to D_ZY' for some Z-reflexive object Y', the second isomorphism implies that

$$f^!Y \cong D_Wf^*D_ZY. \tag{3.4}$$

These observations and the classical context suggest the following definition.

Definition 3.5. A *dualizing object* for a full subcategory $\mathscr{C}_0$ of $\mathscr{C}$ is an object W of $\mathscr{C}$ such that if $X \in \mathscr{C}_0$, then D_WX is in $\mathscr{C}_0$ and X is W-reflexive. Thus D_W specifies an auto–duality of the category $\mathscr{C}_0$.

Remark 3.6. In algebraic geometry, we often encounter canonical subcategories $\mathscr{C}_0 \subset \mathscr{C}$ and $\mathscr{D}_0 \subset \mathscr{D}$ such that $f_!\mathscr{C}_0 \subset \mathscr{D}_0$ and $f^!\mathscr{D}_0 \subset \mathscr{C}_0$ together with a dualizing object Z for $\mathscr{D}_0$ such that $W = f^!Z$ is a dualizing object for $\mathscr{C}_0$. In such contexts, (3.3) and (3.4) express $f_!$ on $\mathscr{C}_0$ and $f^!$ on $\mathscr{D}_0$ in terms of f_* and f^*.

For any objects Y and Z of $\mathscr{D}$, the adjoint of the map

$$f_!(f^*Y \otimes f^!Z) \cong Y \otimes f_!f^!Z \xrightarrow{\mathrm{id}\otimes\sigma} Y \otimes Z$$

is a natural map

$$\phi\colon\, f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z). \tag{3.7}$$

It specializes to

$$\phi\colon\, f^*Y \otimes f^!T \longrightarrow f^!Y, \tag{3.8}$$

which of course compares a right adjoint to a shift of a left adjoint. A *Verdier–Grothendieck isomorphism theorem* asserts that the map ϕ is an isomorphism; in the context of sheaves over spaces, such a result was announced by Verdier in [25, §5]. The following observation, abstracts a result of Neeman [22, 5.4]. In it, we only assume the projection formula for dualizable Y.

Proposition 3.9. *The map $\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$ is an isomorphism for all objects Z and all dualizable objects Y.*

Proof. Using Proposition 1.10, the projection formula, duality adjunctions (1.9), and the $(f_!, f^!)$ adjunction, we obtain isomorphisms

$$\begin{aligned}\mathscr{C}(X, f^*Y \otimes f^!Z) &\cong \mathscr{C}(f^*DY \otimes X, f^!Z) \cong \mathscr{D}(f_!(f^*DY \otimes X), Z)\\ &\cong \mathscr{D}(DY \otimes f_!X, Z) \cong \mathscr{D}(f_!X, Y \otimes Z) \cong \mathscr{C}(X, f^!(Y \otimes Z)).\end{aligned}$$

Diagram chasing shows that the composite isomorphism is induced by ϕ. □

It is natural to ask when ϕ is an isomorphism in general, and we shall return to that question in the context of triangulated categories. Of course, this discussion specializes and remains interesting in the Grothendieck context $f_! = f_*$.

We give some elementary examples of the Verdier–Grothendieck context.

Example 3.10. An example of the Verdier–Grothendieck context is already available with $\mathscr{C} = \mathscr{D}$ and $f^* = f_* = \mathrm{Id}$. Fix an object C of $\mathscr{C}$ and set

$$f_!X = X \otimes C \quad \text{and} \quad f^!(Y) = \mathrm{Hom}(C, Y).$$

The projection formula $f_!(f^*Y \otimes Z) \cong Y \otimes f_!Z$ is the associativity isomorphism

$$(Y \otimes Z) \otimes C \cong Y \otimes (Z \otimes C).$$

The map $\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$ is the canonical map

$$\nu\colon Y \otimes \mathrm{Hom}(C, Z) \longrightarrow \mathrm{Hom}(C, Y \otimes Z).$$

It is an isomorphism if Y is dualizable, and it is an isomorphism for all Y if and only if C is dualizable.

The shift of an adjunction by an object of $\mathscr{C}$ used in the previous example generalizes to give a shift of any Verdier-Grothendieck context by an object of $\mathscr{C}$.

Definition 3.11. For an adjoint pair $(f_!, f^!)$ and an object $C \in \mathscr{C}$, define the twisted adjoint pair $(f_!^C, f_C^!)$ by

$$f_!^C(X) = f_!(X \otimes C) \quad \text{and} \quad f_C^!Y = \mathrm{Hom}(C, f^!Y). \tag{3.12}$$

Proposition 3.13. *If (f^*, f_*) and $(f_!, f!)$ are in the Verdier-Grothendieck context, then so are (f^*, f_*) and $(f_!^C, f!_C)$.*

Proof. The isomorphism $\hat{\pi}$ of (2.3) shifts to a corresponding isomorphism $\hat{\pi}_C$. $\square$

We also give a simple example of the context of Definition 3.5. Recall that dualizable objects are S-reflexive, but not conversely in general. The following observation parallels part of a standard characterization of "dualizing complexes" [11, V.2.1]. Let $d\mathscr{C}$ denote the full subcategory of dualizable objects of $\mathscr{C}$.

Proposition 3.14. *S is W-reflexive if and only if all $X \in d\mathscr{C}$ are W-reflexive.*

Proof. Since S is dualizable, the backwards implication is trivial. Assume that S is W-reflexive. Since $W \cong D_W S$, $\mathrm{Hom}(W, W) = D_W W \cong D_W D_W S$. In any closed symmetric monoidal category, such as $\mathscr{C}$, we have a natural isomorphism

$$\mathrm{Hom}(X \otimes X', X'') \cong \mathrm{Hom}(X, \mathrm{Hom}(X', X'')),$$

where X, X', and X'' are arbitrary objects. When X is dualizable,

$$\nu\colon DX \otimes X' \longrightarrow \mathrm{Hom}(X, X')$$

is an isomorphism for any object X'. Therefore

$$D_W D_W X \cong \mathrm{Hom}(DX \otimes W, W) \cong \mathrm{Hom}(DX, \mathrm{Hom}(W, W)) \cong DDX \otimes D_W D_W S.$$

Identifying X with $X \otimes S$, is easy to check that ρ_W corresponds under this isomorphism to $\rho_S \otimes \rho_W$. The conclusion follows. $\square$

Corollary 3.15. *Let W be dualizable. Then the following are equivalent.*

(i) *W is a dualizing object for $d\mathscr{C}$.*
(ii) *S is W-reflexive.*
(iii) *W is invertible.*
(iv) *$D_W : d\mathscr{C}^{op} \longrightarrow d\mathscr{C}$ is an auto–duality of $d\mathscr{C}$.*

Proof. If X is dualizable, then $D_W X \cong DX \otimes W$ is dualizable. The proposition shows that (i) and (ii) are equivalent, and it is clear that (iii) and (iv) are equivalent. Since W is dualizable, $D_W D_W S \cong \mathrm{Hom}(W, W) \cong W \otimes DW$, with ρ_W corresponding to the coevaluation map $coev : S \longrightarrow W \otimes DW$. By [18, 2.9], W is invertible if and only if $coev$ is an isomorphism. Therefore (ii) and (iii) are equivalent. □

Finally, we have a shift comparison of Grothendieck and Wirthmüller contexts.

Remark 3.16. Start in the Grothendieck context, so that $f_! = f_*$, and assume that the map $\phi\colon f^*Y \otimes f^!T \longrightarrow f^!Y$ of (3.8) is an isomorphism. Assume further that $f^!T$ is invertible and let $C = Df^!T$. Define a new functor $f_!$ by $f_!X = f_*(X \otimes DC)$. Then $f_!$ is *left* adjoint to f^*. Replacing X by $X \otimes C$, we see that

$$f_*X \cong f_!(X \otimes C).$$

In the next section, we shall consider isomorphisms of this general form in the Wirthmüller context. Conversely, start in the Wirthmüller context, so that $f^! = f^*$, and assume given a C such that $f_*S \cong f_!C$ and the map $\omega\colon f_*X \longrightarrow f_!(X \otimes C)$ of (4.7) below is an isomorphism. Define a new functor $f^!$ by $f^!Y = \mathrm{Hom}(C, f^*Y)$ and note that $f^!T \cong DC$. Then $f^!$ is *right* adjoint to f_*. If either C or Y is dualizable, then $\mathrm{Hom}(C, f^*Y) \cong f^*Y \otimes DC$ and thus $f^*Y \otimes f^!T \cong f^!Y$, which is an isomorphism of the same form as in the Grothendieck context.

4. The Wirthmüller isomorphism

We place ourselves in the Wirthmüller context in this section, with $f^! = f^*$. Here the specialization of the Verdier–Grothendieck isomorphism is of no interest. In fact, ϕ reduces to the originally assumed isomorphism (1.1). However, there is now a candidate for an isomorphism between the right adjoint f_* of f^* and a shift of the left adjoint $f_!$. This is not motivated by duality questions, and it can already fail on dualizable objects. We assume in addition to the isomorphisms $\alpha = \bar{\delta}$, hence $\bar{\pi}$ and $\bar{\gamma}$, that we are given an object $C \in \mathscr{C}$ together with an isomorphism

$$f_*S \cong f_!C. \tag{4.1}$$

Observe that the isomorphism $\bar{\gamma}$ specializes to an isomorphism

$$Df_!X \cong f_*DX. \tag{4.2}$$

Taking $X = S$ in (4.2) and using that $DS \cong S$, we see that (4.1) is equivalent to

$$Df_!S \cong f_!C. \tag{4.3}$$

This version is the one most naturally encountered in applications, since it makes no reference to the right adjoint f_* that we seek to understand. In practice, $f_!S$ is dualizable and C is dualizable or even invertible. It is a curious feature of our discussion that it does not require such hypotheses.

Replacing C by $S \otimes C$ in (4.1), it is reasonable to hope that it continues to hold with S replaced by a general X. That is, we can hope for a natural isomorphism

$$f_*X \cong f_\sharp X, \quad \text{where} \quad f_\sharp X \equiv f_!(X \otimes C). \tag{4.4}$$

Note that we twist by C before applying $f_!$. We shall shortly define a particular natural map $\omega\colon f_*X \longrightarrow f_\sharp X$. A *Wirthmüller isomorphism theorem* asserts that ω is an isomorphism. We shall show that if $f_!S$ is dualizable and X is a retract of some f^*Y, then ω is an isomorphism. However, even for dualizable X, ω need *not* be an isomorphism in general. A counterexample is given in the sequel [20]. We

shall also give a categorical criterion for ω to be an isomorphism for a particular object X. An application is also given in [20].

Using the map $T \longrightarrow f_*S$ of (1.2), the assumed isomorphism $f_*S \cong f_!C$ gives rise to maps

$$\tau\colon T \longrightarrow f_*S \cong f_!C$$

and

$$\xi\colon f^*f_!C \cong f^*f_*S \xrightarrow{\varepsilon} S$$

such that

$$\xi \circ f^*\tau = \mathrm{id}\colon S \longrightarrow S.$$

Using the alternative defining property (4.3) of C, we can obtain alternative descriptions of these maps that avoid reference to the functor f_* we seek to understand.

Lemma 4.5. *The maps τ and ξ coincide with the maps*

$$T \cong DT \xrightarrow{D\sigma} Df_!f^*T \cong Df_!S \cong f_!C$$

and

$$f^*f_!C \cong f^*Df_!S \cong Df^*f_!S \xrightarrow{D\zeta} DS \cong S.$$

Proof. The isomorphism $Df_!S \cong f_!C$ used in the displays is the composite of the given isomorphism (4.1) and the special case (4.2) of the isomorphism $\bar{\gamma}$. The proofs are diagram chases that use the naturality of η and ε, the triangular identities for the $(f_!, f^*)$ adjunction, and the description of $\bar{\gamma}$ in Proposition 2.11. $\square$

Using the isomorphism (2.7), we extend τ to the natural map

$$\tau\colon Y \cong Y \otimes T \xrightarrow{\mathrm{id}\otimes\tau} Y \otimes f_!C \cong f_!(f^*Y \otimes C) = f_\sharp f^*Y. \tag{4.6}$$

Specializing to $Y = f_*X$, we obtain the desired comparison map ω as the composite

$$\omega\colon f_*X \xrightarrow{\tau} f_\sharp f^*f_*X \xrightarrow{f_\sharp\varepsilon} f_\sharp X. \tag{4.7}$$

An easy diagram chase using the triangular identity $\varepsilon \circ f^*\eta = \mathrm{id}$ shows that

$$\omega \circ \eta = \tau\colon Y \longrightarrow f_\sharp f^*Y. \tag{4.8}$$

If ω is an isomorphism, then τ must be the unit of the resulting $(f^*, f_\sharp)$ adjunction.

Similarly, using (1.1) and (2.7), we extend ξ to the natural map

$$\xi\colon f^*f_\sharp f^*Y = f^*f_!(f^*Y \otimes C) \cong f^*Y \otimes f^*f_!C \xrightarrow{\mathrm{id}\otimes\xi} f^*Y \otimes S \cong f^*Y. \tag{4.9}$$

We view ξ as a partial counit, defined not for all X but only for $X = f^*Y$. Since $\xi \circ f^*\tau = \mathrm{id}\colon S \longrightarrow S$, it is immediate that

$$\xi \circ f^*\tau = \mathrm{id}\colon f^*Y \longrightarrow f^*Y, \tag{4.10}$$

which is one of the triangular identities for the desired $(f^*, f_\sharp)$ adjunction. Define

$$\psi\colon f_\sharp f^*Y \longrightarrow f_*f^*Y \tag{4.11}$$

to be the adjoint of ξ. The adjoint of the relation (4.10) is the analogue of (4.8):

$$\psi \circ \tau = \eta\colon Y \longrightarrow f_*f^*Y. \tag{4.12}$$

Proposition 4.13. *If Y or $f_!S$ is dualizable, then $\omega : f_*f^*Y \longrightarrow f_\sharp f^*Y$ is an isomorphism with inverse ψ. If ψ is an isomorphism for all Y, then $f_!S$ is dualizable. If X is a retract of some f^*Y, where Y or $f_!S$ is dualizable, then $\omega : f_*X \longrightarrow f_\sharp X$ is an isomorphism.*

Proof. With $X = f^*Y$, the first part of the proof of the following result gives that $\psi \circ \omega = \mathrm{id}$, so that $\omega = \psi^{-1}$ when ψ is an isomorphism. We claim that ψ coincides with the following composite:

$$f_\sharp f^*Y = f_!(f^*Y \otimes C) \cong Y \otimes D(f_!S) \xrightarrow{\nu} \mathrm{Hom}(f_!S, Y) \cong f_* \mathrm{Hom}(S, f^*Y) = f^*Y.$$

Here the isomorphisms are given by (2.7) and (4.3) and by (2.5). Since ν is an isomorphism if Y or $f_!S$ is dualizable, the claim implies the first statement. Note that $\psi = f_*\xi \circ \eta$ and that the isomorphism $\bar{\gamma}$ of (2.5) is $f_* \mathrm{Hom}(\zeta, \mathrm{id}) \circ f_*\alpha \circ \eta$. Using the naturality of η and the description of ξ in Lemma 4.5, an easy, if lengthy, diagram chase shows that the diagram (1.8) gives just what is needed to check the claim. The second statement is now clear by the definition of dualizability: it suffices to consider $Y = f_!S$. The last statement follows from the first since a retract of an isomorphism is an isomorphism. $\square$

We extract a criterion for ω to be an isomorphism for a general object X from the usual proof of the uniqueness of adjoint functors [17, p. 85].

Proposition 4.14. *If there is a map* $\xi\colon f^*f_\sharp X = f^*f_!(X \otimes C) \longrightarrow X$ *such that*

$$f_\sharp \xi \circ \tau = \mathrm{id}\colon f_\sharp X \longrightarrow f_\sharp X \tag{4.15}$$

and the following (partial naturality) diagram commutes, then $\omega\colon f_*X \longrightarrow f_\sharp X$ *is an isomorphism with inverse the adjoint* ψ *of* ξ.

$$\begin{tikzcd} f^*f_\sharp f^*f_*X \arrow[r, "\xi"] \arrow[d, "f^*f_\sharp\varepsilon"'] & f^*f_*X \arrow[d, "\varepsilon"] \\ f^*f_\sharp X \arrow[r, "\xi"'] & X \end{tikzcd} \tag{4.16}$$

Moreover, (4.15) holds if and only if the following diagram commutes.

$$\begin{tikzcd} X \otimes C \arrow[r, "\zeta"] \arrow[d, "\zeta"'] & f^*f_!(X \otimes C) \\ f^*f_!(X \otimes C) \arrow[r, "f^*\tau"'] & f^*f_!(f^*f_!(X \otimes C) \otimes C) \arrow[u, "f^*f_!(\xi\otimes\mathrm{id})"'] \end{tikzcd} \tag{4.17}$$

Proof. In the diagram (4.16), the top map ξ is given by (4.9). The diagram and the relation $\xi \circ f^*\tau = \mathrm{id}$ of (4.10) easily imply the relation $\xi \circ f^*\omega = \varepsilon$, which is complementary to the defining relation $\varepsilon \circ f^*\psi = \xi$ for the adjoint ψ. Passage to adjoints gives that $\psi \circ \omega = \mathrm{id}$. The following diagram commutes by (4.8), the triangular identity $f_*\varepsilon \circ \eta = \mathrm{id}$, the naturality of η and ω, and the fact that ψ is adjoint to ξ. It gives that $\omega \circ \psi = f_\sharp \xi \circ \tau = \mathrm{id}$.

$$\begin{tikzcd}
f_\sharp X \arrow[rrr, "\tau"] \arrow[dr, "\eta"] \arrow[ddd, "\psi"'] & & & f_\sharp f^* f_\sharp X \arrow[ddd, "f_\sharp \xi"] \arrow[dl, equal] \\
& f_* f^* f_\sharp X \arrow[r, "\omega"] \arrow[d, "f_* f^* \psi"'] & f_\sharp f^* f_\sharp X \arrow[d, "f_\sharp f^* \psi"] & \\
& f_* f^* f_* X \arrow[r, "\omega"] \arrow[d, "f_* \varepsilon"] & f_\sharp f^* f_* X \arrow[dr, "f_\sharp \varepsilon"] & \\
f_* X \arrow[r, equal] \arrow[ur, "\eta"] & f_* X \arrow[rr, "\omega"'] & & f_\sharp X.
\end{tikzcd}$$

The last statement is clear by adjunction. □

Remark 4.18. The map ω can be generalized to the Verdier–Grothendieck context. For that, we assume given an object W of $\mathscr{C}$ such that

$$f_! C \cong D f_! f^! T;$$

compare (4.3). As in Lemma 4.5, we then have the map

$$\tau\colon T \cong DT \xrightarrow{D\sigma} D f_! f^! T \cong f_! C.$$

This allows us to define the comparison map

$$\omega\colon f_* X \cong f_* X \otimes T \xrightarrow{\mathrm{id}\otimes\tau} f_* X \otimes f_! W \cong f_!(f^* f_* X \otimes C) \xrightarrow{f_!(\varepsilon\otimes\mathrm{id})} f_!(X \otimes C.$$

A study of when this map ω is an isomorphism might be of interest, but we have no applications in mind. We illustrate the idea in the context of Example 3.10.

Example 4.19. Returning to Example 3.10, we seek an object C' of $\mathscr{C}$ such that $f_! C' \cong D(f_! f^! S)$, which is

$$C' \otimes C \cong D(DC \otimes C).$$

If C is dualizable, then the right side is isomorphic to $C \otimes DC \cong DC \otimes C$ and we can take $C' = DC$. Here the map

$$\omega\colon X = f_* X \longrightarrow f_!(X \otimes DC) = X \otimes DC \otimes C$$

turns out to be $\mathrm{id} \otimes (\gamma \circ coev)$, where $coev\colon S \longrightarrow C \otimes DC$ is the coevaluation map of the duality adjunction (1.9) and γ is the commutativity isomorphism for $\otimes$. We conclude (e.g., by [18, 2.9]) that ω is an isomorphism if and only if C is invertible.

5. Preliminaries on triangulated categories

We now go beyond the hypotheses of §§1–4 to the triangulated category situations that arise in practice. We assume that $\mathscr{C}$ and $\mathscr{D}$ are triangulated and that the functors $(-) \otimes X$ and f^* are exact (or triangulated). This means that they are additive, commute up to isomorphism with Σ, and preserve distinguished triangles. For $(-) \otimes X$, this is a small part of the appropriate compatibility conditions that relate distinguished triangles to $\otimes$ and Hom in well-behaved triangulated closed symmetric monoidal categories; see [19] for a discussion of this, as well as for basic observations about what triangulated categories really are: the standard axiom system is redundant and unnecessarily obscure. We record the following easily proven observation relating adjoints to exactness (see for example [21, 3.9]).

Lemma 5.1. *Let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G : \mathscr{B} \longrightarrow \mathscr{A}$ be left and right adjoint functors between triangulated categories. Then F is exact if and only if G is exact.*

We also record the following definitions (see for example [12, 22]).

Definition 5.2. A full subcategory $\mathscr{B}$ of a triangulated category $\mathscr{C}$ is *thick* if any retract of an object of $\mathscr{B}$ is in $\mathscr{B}$ and if the third object of a distinguished triangle with two objects in $\mathscr{B}$ is also in $\mathscr{B}$. The category $\mathscr{B}$ is *localizing* if it is thick and closed under coproducts. The smallest thick (respectively, localizing) subcategory of $\mathscr{C}$ that contains a set of objects $\mathscr{G}$ is called the thick (respectively, localizing) subcategory generated by $\mathscr{G}$.

Definition 5.3. An object X of an additive category $\mathscr{A}$ is *compact*, or *small*, if the functor $\mathscr{A}(X,-)$ converts coproducts to direct sums. The category $\mathscr{A}$ is *compactly generated* if it has arbitrary coproducts and has a set $\mathscr{G}$ of compact objects that detects isomorphisms, in the sense that a map f in $\mathscr{A}$ is an isomorphism if and only if $\mathscr{A}(X,f)$ is an isomorphism for all $X \in \mathscr{G}$. When $\mathscr{A}$ is symmetric monoidal, we require its unit object to be compact; thus it can be included in the set $\mathscr{G}$.

In the triangulated case, this is equivalent to Neeman's definition [22, 1.7]. With our version, we have the following generalization of a result of his [22, 5.1].

Lemma 5.4. *Let $\mathscr{A}$ be a compactly generated additive category with generating set $\mathscr{G}$ and let $\mathscr{B}$ be any additive category. Let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor with right adjoint G. If G preserves coproducts, then F preserves compact objects. Conversely, if $F(X)$ is compact for $X \in \mathscr{G}$, then G preserves coproducts.*

Proof. Let $X \in \mathscr{A}$ and let $\{Y_i\}$ be a set of objects of $\mathscr{B}$. Then the evident map $f : \amalg G(Y_i) \longrightarrow G(\amalg Y_i)$ induces a map

$$f_*\colon \mathscr{A}(X, \amalg G(Y_i)) \longrightarrow \mathscr{A}(X, G(\amalg Y_i)).$$

If X is compact and f_* is an isomorphism, then, by adjunction and compactness, it induces an isomorphism

$$\amalg \mathscr{B}(F(X), Y_i) \longrightarrow \mathscr{B}(F(X), \amalg Y_i),$$

which shows that $F(X)$ is compact. Conversely, if X and $F(X)$ are both compact, then f_* corresponds under adjunction to the identity map of $\amalg \mathscr{B}(F(X), Y_i)$ and is therefore an isomorphism. Restricting to $X \in \mathscr{G}$, it follows from Definition 5.3 that f is an isomorphism. $\square$

While this result is elementary, it is fundamental to the applications. We generally have much better understanding of left adjoints, so that the compactness criterion is verifiable, but it is the preservation of coproducts by right adjoints that is required in all of the formal proofs.

Returning to triangulated categories, we justify the term "generating set" by the following result. Its first part is [22, 3.2], and its second part is [12, 2.1.3(d)].

Proposition 5.5. *Let $\mathscr{A}$ be a compactly generated triangulated category with generating set $\mathscr{G}$. Then the localizing subcategory generated by $\mathscr{G}$ is $\mathscr{A}$ itself. If the objects of $\mathscr{G}$ are dualizable, then the thick subcategory generated by $\mathscr{G}$ is the full subcategory of dualizable objects in $\mathscr{A}$, and an object is dualizable if and only if it is compact.*

The following standard observation works in tandem with the previous result.

Proposition 5.6. *Let $F, F' \colon \mathscr{A} \longrightarrow \mathscr{B}$ be exact functors between triangulated categories and let $\phi \colon F \longrightarrow F'$ be a natural transformation that commutes with Σ. Then the full subcategory of $\mathscr{A}$ whose objects are those X for which ϕ is an isomorphism is thick, and it is localizing if F and F' preserve coproducts.*

Proof. Since a retract of an isomorphism is an isomorphism, closure under retracts is clear. Closure under triangles is immediate from the five lemma. A coproduct of isomorphisms is an isomorphism, so closure under coproducts holds when F and F' preserve coproducts. □

6. The formal isomorphism theorems

We assume throughout this section that $\mathscr{C}$ and $\mathscr{D}$ are closed symmetric monoidal categories with compatible triangulations and that (f^*, f_*) is an adjoint pair of functors with f^* strong symmetric monoidal and exact.

For the Wirthmüller context, we assume in addition that f^* has a left adjoint $f_!$. The maps (2.7)–(2.6) are then given by (1.3) and Proposition 2.11. When

$$\bar{\pi} \colon f_!(f^*Y \otimes X) \longrightarrow Y \otimes f_!X$$

is an isomorphism, the map

$$\omega : f_*X \longrightarrow f_!(X \otimes C)$$

is defined. Observe that $\bar{\pi}$ is a map between exact left adjoints and that $\bar{\pi}$ and ω commute with Σ. The results of the previous section give the following conclusion.

Theorem 6.1 (Formal Wirthmüller isomorphism). *Let $\mathscr{C}$ be compactly generated with a generating set $\mathscr{G}$ such that $\bar{\pi}$ and ω are isomorphisms for $X \in \mathscr{G}$. Then $\bar{\pi}$ is an isomorphism for all $X \in \mathscr{C}$. If the objects of $\mathscr{G}$ are dualizable, then ω is an isomorphism for all dualizable X. If f^*X is compact for $X \in \mathscr{G}$, then ω is an isomorphism for all $X \in \mathscr{C}$.*

The force of the theorem is that no construction of an inverse to ω is required: we need only check that ω is an isomorphism one generating object at a time. Proposition 4.14 explains what is needed for that verification.

For the Grothendieck context, we can use the following basic results of Neeman [22, 3.1, 4.1] to construct the required right adjoint $f^!$ to f_* in favorable cases. A main point of Neeman's later monograph [23] and of Franke's paper [9] is to replace compact generation by a weaker notion that makes use of cardinality considerations familiar from the theory of Bousfield localization in algebraic topology.

Theorem 6.2 (Triangulated Brown representability theorem). *Let $\mathscr{A}$ be a compactly generated triangulated category. A functor $H \colon \mathscr{A}^{op} \longrightarrow \mathscr{A}b$ that takes distinguished triangles to long exact sequences and converts coproducts to products is representable.*

Theorem 6.3 (Triangulated adjoint functor theorem). *Let $\mathscr{A}$ be a compactly generated triangulated category and $\mathscr{B}$ be any triangulated category. An exact functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ that preserves coproducts has a right adjoint G.*

Proof. Take $G(Y)$ to be the object that represents the functor $\mathscr{B}(F(-), Y)$. □

The map

$$\pi \colon Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X)$$

of (1.6) commutes with Σ. When π is an isomorphism,

$$\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$$

is defined and commutes with Σ. We obtain the following conclusion.

Theorem 6.4 (Formal Grothendieck isomorphism)**.** *Let $\mathscr{D}$ be compactly generated with a generating set $\mathscr{G}$ such that f^*Y is compact and π is an isomorphism for $Y \in \mathscr{G}$. Then f_* has a right adjoint $f^!$, π is an isomorphism for all $Y \in \mathscr{D}$, and ϕ is an isomorphism for all dualizable Y. If the functor $f^!$ preserves coproducts, then ϕ is an isomorphism for all $Y \in \mathscr{D}$.*

Proof. As a right adjoint of an exact functor, f_* is exact by Lemma 5.1, and it preserves coproducts by Lemma 5.4. Thus $f^!$ exists by Theorem 6.3. Now π is an isomorphism for all Y by Proposition 5.6, ϕ is an isomorphism for dualizable Y by Proposition 3.9, and the last statement holds by Propositions 5.5 and 5.6. $\square$

When $f^!$ is obtained abstractly from Brown representability, the only sensible way to check that it preserves coproducts is to appeal to Lemma 5.4, requiring $\mathscr{C}$ to be compactly generated and f_*X to be compact when X is in the generating set.

For the Verdier-Grothendieck context, we assume that we have a second adjunction $(f_!, f^!)$, with $f_!$ exact. We also assume given a map

$$\hat{\pi}\colon Y \otimes f_!X \cong f_!(f^*Y \otimes X)$$

that commutes with Σ. When $\hat{\pi}$ is an isomorphism, the map

$$\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$$

is defined and commutes with Σ. Using Proposition 3.9 and the results of the previous section, we obtain the following conclusion.

Theorem 6.5 (Formal Verdier isomorphism)**.** *Let $\mathscr{D}$ be compactly generated with a generating set $\mathscr{G}$ such that f^*Y is compact and $\hat{\pi}$ is an isomorphism for $Y \in \mathscr{G}$. Then $\hat{\pi}$ is an isomorphism for all $Y \in \mathscr{D}$, and ϕ is an isomorphism for all dualizable Y. If the functor $f^!$ preserves coproducts, then ϕ is an isomorphism for all $Y \in \mathscr{D}$.*

Remark 6.6. In many cases, one can construct a more explicit right adjoint $f_0^!$ from some subcategory $\mathscr{D}_0$ of $\mathscr{D}$ to some subcategory $\mathscr{C}_0$ of $\mathscr{C}$, as in Remark 3.6. In such cases we can combine approaches. Indeed, assume that we have an adjoint pair $(f_!, f_0^!)$ on full subcategories $\mathscr{C}_0$ and $\mathscr{D}_0$ such that objects isomorphic to objects in $\mathscr{C}_0$ (or $\mathscr{D}_0$) are in $\mathscr{C}_0$ (or $\mathscr{D}_0$). Then, by the uniqueness of adjoints, the right adjoint $f^!$ to $f_!$ given by Brown representability restricts on $\mathscr{D}_0$ to a functor with values in $\mathscr{C}_0$ that is isomorphic to the explicitly constructed functor $f_0^!$. That is, the right adjoint given by Brown representability can be viewed as an extension of the functor $f_0^!$ to all of $\mathscr{D}$. This allows quotation of Proposition 2.4 or 2.9 for the construction and comparison of the natural maps (2.3)–(2.2).

We give an elementary example and then some remarks on the proofs of the results that we have quoted from the literature, none of which are difficult.

Example 6.7. Return to Example 3.10, but assume further that $\mathscr{C}$ is a compactly generated triangulated category. Here the formal Verdier duality theorem says that $\phi = \nu : Y \otimes \mathrm{Hom}(C, Z) \longrightarrow \mathrm{Hom}(C, Y \otimes Z)$ is an isomorphism if and only if the functor $\mathrm{Hom}(C, -)$ preserves coproducts. That is, an object C is dualizable if and only if $\mathrm{Hom}(C, -)$ preserves coproducts.

Remark 6.8. Clearly Theorem 6.3 is a direct consequence of Theorem 6.2. In turn, Theorem 6.2 is essentially a special case of Brown's original categorical representation theorem [3]. Neeman's self-contained proof closely parallels Brown's argument. The first statement of Proposition 5.5 is used as a lemma in the proof, but it is also a special case. To see this, let $\mathscr{B}$ be the localizing subcategory of $\mathscr{A}$ generated by $\mathscr{G}$. Then, applied to the functor $\mathscr{A}(-,X)$ on $\mathscr{B}$ for $X \in \mathscr{A}$, the representability theorem gives an object $Y \in \mathscr{B}$ and an isomorphism $Y \cong X$ in $\mathscr{A}$. The second part of Proposition 5.5 is intuitively clear, since objects in $\mathscr{A}$ not in the thick subcategory generated by $\mathscr{G}$ must involve infinite coproducts, and these will be neither dualizable nor compact. The formal proof in [12] starts from Example 6.7, which effectively ties together dualizability and compactness.

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