Algebraic K-Theory and Topological Spaces

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Abstract. In this note we discuss the algebraic and topological K-theories of an admissible space X and demonstrate how one may recover $\boldsymbol{ku}^*(X)$ the connective topological K-theory of X from the algebraic K-theory $K_*^B(\Delta^\bullet \times X)$. By considering Thomason's formulation of hypercohomology with coefficients in a presheaf of spectra, we present a new look at the exponential sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^* \to 0,$$

and relate the cycle class map to K-theory.

Mathematics Subject Classifications: 14C35, 19D20, 55N30

This paper is dedicated to the memory of R. W. Thomason.

Introduction

In the course of his inquiries into several profound problems in mathematics, the acute insight of R. W. Thomason invariably led, in addition to elegant solutions, revolutionary advances. In this note we shall see how a few of Thomason's ideas lead to surprising connections between the diverse areas of algebraic and topological K-theories. For a brief survey of the work of Thomason see [53].

Given a smooth complex variety X, let X^{an} denote the corresponding complex analytic space. If E is an algebraic vector bundle over X, the associated analytic space E^{an} is a topological vector bundle over X^{an}. In fact there is a natural transformation

$$(\tau_X)_* : K_*(X) \to \boldsymbol{ku}^{-*}(X^{an})$$

of graded abelian groups, where $K_*(X)$ is the algebraic K-theory of X and $\boldsymbol{ku}^{-*}(X^{an})$ is the connective K-theory of X^{an}. For n sufficiently large the universal chern character

$$\mathrm{ch}_i \in H^{2i}(\mathrm{BGL_n}(\mathbf{C}); \mathbf{Q});$$

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moreover, $H^{2i}(\mathrm{BGL_n}(\mathbf{C});\mathbf{Q})$ has a pure hodge structure of weight $2i$ with $H^{2i}(\mathrm{BGL_n}(\mathbf{C});\mathbf{C}) = H^{(i,i)}(\mathrm{BGL}_n(\mathbf{C}))$. Thus (cf. [31, p. 90])

$$\mathrm{ch}_i\big(K_j(X)\otimes\mathbf{Q}\big) \subseteq W_{2i}H^{2i-j}(X^{an};\mathbf{Q}) \cap F^iH^{2i-j}(X^{an};\mathbf{C}).$$

It follows that if X is projective $\mathrm{ch}_i\big(K_j(X)\otimes\mathbf{Q}\big) = 0$, hence $(\tau_X)_j$ for $j > 0$, has torsion image. Nevertheless the transformation τ provides a key to the algebro-geometric substructure of X^{an}. Indeed, let $\mathcal{K}_p$ denote the sheaf in the Zariski topology associated to the presheaf $U \mapsto K_p\big(\mathcal{O}(U)\big)$; similarly let $\mathcal{K}_p^{an}$ denote the sheaf in the classical topology associated to $U \mapsto \mathcal{O}^{an}(U)$. There is a spectral sequence

$$E_1^{p,q} = H^p(X;\mathcal{K}_{-q}) \Longrightarrow K_{-p-q}(X)$$

and $H^p(X;\mathcal{K}_p) \simeq CH^p(X)$, where $CH^p(X)$ is the Chow group of codimension p-cycles on X. For $p = 1$ the sheaf $\mathcal{K}_1 \simeq \mathcal{O}^*$, and the exponential sequence takes the form

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{K}_1^{an} \to 0$$

on X^{an} giving a sheaf theoretic formulation of the familiar cycle class map

$$CH^1(X) \simeq H^1(X^{an};\mathcal{K}_1^{an}) \to H^2(X^{an};\mathbf{Z})$$

Also note that the Brown-Gersten-Quillen spectral sequence above asserts that $CH^1(X)$ is a subquotient of $K_0(X)$, and by identifying $\mathbf{Z} \simeq \boldsymbol{ku}^{-2}$ the Atiyah-Hirzebruch spectral sequence asserts that $H^2(X^{an};\mathbf{Z})$ is a subquotient of $\boldsymbol{ku}^0(X^{an})$.

In [6] Bloch describes an extension of $\mathcal{K}_2$ by $\mathcal{K}_1/\Delta^*$, where $\Delta^* \subset \mathbf{C}$ is a countable subgroup. In this note the reader will find a second interpertation and generalization of the exponential sequence. From the view point of homotopy theory, the exponential map is a universal cover and $\pi_1\mathbf{C}^* \simeq \boldsymbol{ku}^{-2} \simeq \mathbf{Z}$ with a preferred generator $t \mapsto \exp(2\pi\sqrt{-1}t)$. By declaring $\det : K_1(\mathbf{C}) \to \mathbf{C}^*$ to be a homeomorphism, we obtain a topological structure on $K_1(\mathbf{C})$. Unfortunately, this observation does not indicate that $K_q(\mathbf{C})$, for $q > 1$, might have an interesting topological structure. To proceed further we construct simplicial abelian groups $K_q\big(\mathbf{C}(\Delta^\bullet)\big)$ whose homotopy groups $\pi_pK_q\big(\mathbf{C}(\Delta^\bullet)\big)$ form the E^2-terms of a spectral sequence convegring to $\boldsymbol{ku}^{-p-q}$. Rather than considerinf extensions of $K_q(\mathbf{C})$ by $K_{q-1}(\mathbf{C})$ as in *op.cit.*, we associate to $K_q(\mathbf{C})$ an increasing sequence of higher connected covers. To be specific we construct, in a manner dual to the Postnikov tower, a Whitehead tower

$$\cdots \to W_2 \to W_1 \to K_q\big(\mathbf{C}(\Delta^\bullet)\big)$$

of fibrations with

$$\pi_p W_n = \begin{cases} \pi_p K_q(\mathbf{C}(\Delta^\bullet)) & \text{if } p > n \\ 0 & \text{if } p \leq n, \end{cases}$$

and the fiber of $W_n \to W_{n-1}$ is an Eilenberg-Mac Lanes space of type $K(\pi_n K_q(\mathbf{C}(\Delta^\bullet)), n-1)$ generalizing the exponential sequence. This tower is readily deployed to form a spectral sequence converging to the connective K-theory of X^{an}.

In §1 we indicate how one can extend the well known Serre-Swan correspondence between vector bundles over a compact space X and finitely generated projective modules over the ring $\mathbf{C}(X)$ of continuous function on X to the larger class of admissible spaces. This result is not new cf. [49], but does not seem to be well known. With this result in hand we are able to make a fairly simple construction via infinite loop space machinery of a connective spectrum $K^{\mathrm{top}}(\mathbf{C}(X))$ whose homotopy groups are the topological K-groups of X in negative cohomological dimensions, i.e., for $p \geq 0$

$$\pi_p K^{\mathrm{top}}(\mathbf{C}(X)) \cong \boldsymbol{ku}^{-p}(X) = [\Sigma^p X_+, \mathbf{Z} \times BU]_o,$$

where for pointed spaces A and B we denote the set of pointed homotopy classes of maps from A into B by $[A, B]_o$. We then compare the algebraic K-theory of the discrete ring $\mathbf{C}(X)^\delta$ with the topological K-theory of the topological ring $\mathbf{C}(X)^{\mathrm{top}}$. By replacing the topological ring $\mathbf{C}(X)^{\mathrm{top}}$ with the simplicial ring $\mathbf{C}(\Delta^\bullet \times X)^\delta$ we obtain a spectral sequence starting from the algebraic K-theory of $\mathbf{C}(\Delta^p \times X)^\delta$ and converging to the topological K-theory of $\mathbf{C}(X)^{\mathrm{top}}$. We turn our attention in §2 to recovering the connective topological K-theory of X, i.e. $\boldsymbol{ku}^p(X)$ for $p > 0$. Borrowing techniques from [48] we extend the connective spectrum $K(\mathbf{C}(X))$ to a non-connective spectrum $K^B(\mathbf{C}(X))$, and we show (see 2.7) that there is a natural isomorphism

$$K^B H_*^{cont}(X) \cong \boldsymbol{ku}^{-*}(X)$$

of graded groups. J. Rosenberg [41] first signaled the relationship between integral negative algebraic K-theory and $\boldsymbol{ku}$.

In §3 we work locally and consider hypercohomology with coefficients is a presheaf of simplicial spectra $\mathcal{E}_\bullet$. We formulate a spectral sequence, based on the seminal work of Thomason [47], relating the hypercohomology with coefficients in the simplicial sheaf $\tilde{\pi}_t \mathcal{E}_\bullet$ to the hypercohomology with coefficients in $|\mathcal{E}_\bullet|$. The last section translates the familiar exponential sequence into a statement about hypercohomology with coefficients in simplicial presheaves of spectra. We include

a brief discussion of spectra in an appendix, and we give a construction of a pairing for spectral sequences of simplicial spectra.

We assume the reader is familiar with the concept of a category with cofibrations and weak equivalences, and the construction, as described in [48], of a K-theory spectrum from such objects.

1. Algebraic methods in topological K-theory

1.1. Admissible spaces

Let X be a topological space. Following Hirzebruch [24] we say that X is an *admissible* space if it is Hausdorff, locally compact, a countable union of compact subsets, and of finite topological dimension. Recall that a topological space X is said to be *finite-dimensional* if there is some non-negative integer n such that every open covering $\mathcal{A}$ of X has a refinement $\mathcal{B}$ with the property that each point $x \in X$ is contained in at most $n+1$ elements of $\mathcal{B}$, and X is said to have *topological dimension* n if n is the smallest integer for which this property holds (cf. [37, §7.9]). Since locally compact spaces are regular, a theorem of Morita (see [30]) asserts that the first two conditions guarantee that admissible spaces are paracompact, hence normal. Moreover a space is admissible if and only if it can be embedded as a closed subset in $\mathbf{R}^N$ for some N.

Consider the ring $\mathbf{C}(X)$ of continuous complex valued functions on an admissible space X. If X is compact, a theorem of Swan [45] establishes an equivalence between finitely generated projective $\mathbf{C}(X)$-modules and complex vector bundles over X. A result of Vaserstein [49] says that this result of Swan holds for a larger class of spaces. Using some well known results [36] and [3] from the theory of vector bundles, we shall see that Swan's result does hold for the class of admissible spaces.

Let X be an admissible space, and let $p : E \to X$ be a complex vector bundle. Denote by $\Gamma(E)$ the $\mathbf{C}(X)$-module of continuous sections of E. Following Atiyah [3, p. 26] we call a $\mathbf{C}(X)$-submodule $V \subseteq \Gamma(E)$ *ample* if the map $\phi : X \times V \to E$ defined by $\phi(x, s) = s(x)$ is surjective.

1.2 Lemma. *If E is a finite dimensional complex vector bundle over an admissible space, then E is generated by a finite number of global sections.*

Proof. Suppose that $\dim X \leq n$. Let $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ be an open covering of X such that (i) each point is contained in at most $n+1$ elements of $\mathcal{V}$, (ii) $E|V_\alpha$ is trivial for each α, and (iii) there is a

partition of unity $\{\lambda_\alpha\}$ subordinate to $\mathcal{V}$. We define, as in [36], for each non-empty finite subset $S \subseteq A$, the set

$$U(S) \subseteq X := \{x \in X : \min_{\alpha \in S} \lambda_\alpha(x) > \max_{\alpha \notin S} \lambda_\alpha(x)\}.$$

Let U_k equal to the union of the subsets $U(S)$ with $\#S = k$. We have that each subset U_k is open, if $k > n+1$ then $U_k = \emptyset$ and $X = U_1 \cup \cdots \cup U_{n+1}$. Since $U(S) \subseteq V_\alpha$ if $\alpha \in S$, $E|U(S)$ is trivial. Furthermore U_k is the disjoint union of its open subsets $U(S)$. Thus $E|U_k$ is also trivial. Let $\{\rho_k\}$ be a partition of unity subordinate to the finite open covering $\{U_k\}$. For each k there exists a finitely generate ample submodule $G_k \subseteq \Gamma(E|U_k)$. Define $\sigma_k : G_k \to \Gamma(E)$ by

$$\sigma_k(s)(x) = \begin{cases} \rho_k(x)s(x) & \text{if } x \in U_k \\ 0 & \text{if } x \notin U_k \end{cases}$$

Now for each $x \in X$ there exists k such that $\rho_k(x) > 0$; in particular the composition $G_k \to \Gamma(E) \to E_x$ is surjective. Thus we obtain a homomorphism $\sigma : \prod G_k \to \Gamma(E)$ whose image is a finitely generated ample submodule. □

This lemma asserts that vector bundles over admissible spaces are very similar to vector bundles over compact spaces. Given this lemma and the general theory of vector bundles [3], the proof of the following corollary is straightforward.

1.3 Corollary. *If E is a finite dimensional complex vector bundle over an admissible space X, then there exist a vector bundle epimorphism $f : X \times \mathbf{C}^n \to E$ and a vector bundle isomorphism $E \oplus \ker(f) \cong X \times \mathbf{C}^n$.*

It is now a simple exercise (cf. [3, §1.4]) to extend Swan's result, which is commonly referred to as the Serre-Swan correspondence, to

1.4 Proposition. *Let X be an admissible space. Then a $\mathbf{C}(X)$-module P is isomorphic to a module of the form $\Gamma(E)$, where $E \to X$ is a finite dimensional complex vector, if and only if P is projective and of finite type.*

In particular the Grothendieck group of complex vector bundles on X is isomorphic to the Grothendieck group of finitely generated projective $\mathbf{C}(X)$-modules.

1.5 Remark. If X is a locally compact space and is a countable union of compact subsets but is not finite dimensional, it is not necessarily true that vector bundles on X are quotients of trivial vector bundles of

finite type. For example the infinite complex projective space $\cup \mathbf{P}^n = \mathbf{P}^\infty$ is such a space, and the universal bundle $E = \{(l, v) \in \mathbf{P}^\infty \times \mathbf{C}^\infty : v \in l\}$ is not the quotient of a trivial vector bundle of finite type since the inverse of the total chern class $1 + c_1$ of E is

$$(1 + c_1)^{-1} = \sum_{n \geq 0} (-1)^n c_1^n \in H^*(\mathbf{P}^\infty; \mathbf{Z}) = \mathbf{Z}[\![c_1]\!].$$

1.6. K-theory spectra

Let $\boldsymbol{ku}$ denote the connective topological K-theory spectrum. Recall that the $\boldsymbol{ku}$ is the connective spectrum associated to the the infinite loop space $\mathbf{Z} \times BU$; in particular the homotopy groups $\boldsymbol{ku}$ are

$$\pi_n \boldsymbol{ku} = \boldsymbol{ku}^{-n} = \begin{cases} \mathbf{Z} & \text{if } n \text{ is even and non-negative} \\ 0 & \text{otherwise.} \end{cases}$$

Let X be an admissible space, and let $A = \mathbf{C}(X)$. The algebra A is a Fréchet algebra. Indeed, if $K_0 \subset K_1 \subset K_2 \subset \ldots$ is a countable sequence of compact subsets such that $X = \cup K_i$, define a countable family of multiplicative semi-norms p_i by $p_i(f) = \sup\{|f(x)| : x \in K_i\}$. The collection $\{p_i\}$ of semi-norms induce the compact-open topology on A, and it follows that A is a complete locally convex algebra whose topology can be defined by a norm. Further details can be found in [23].

Let $\mathsf{Proj}(A)$ denote the category of finitely generated projective A-modules. Given $P \in \mathsf{Proj}(A)$ let $\{p_1, \ldots, p_n\}$ be a finite set of generators, and define the obvious surjective map $\pi : A^n \to P$. We then give P the quotient topology associated to π. It is a simple exercise to show that the topology on P does not depend on the choice of generators; moreover, all maps in $\mathsf{Proj}(A)$ are continuous. Thus $\mathsf{Proj}(A)$ carries an enriched structure of a topological category. Direct sum in $\mathsf{Proj}(A)$ gives it the addition structure of a topological symmetric monoidal category. We denote by $K^{\text{top}}(A)$ the connective spectrum associated to A via Segal's Γ-spaces [42], and we set

$$K_n^{\text{top}}(A) := \pi_n K^{\text{top}}(A).$$

1.7 Remark. We have chosen Segal's infinite loop space machine to produce the connective topological K-theory spectrum of A. By the work of May and Thomason [34, 46], this choice is not important. Segal's machine is singled out because of its relationship to Waldhausen's $S.$-construction [51, §1.8].

1.8. Infinite loop spaces

To get a better grasp of the connective spectrum $K^{\mathrm{top}}(A)$ we summarize the infinite loop machinery of Segal [42]. For a comprehensive study of the relationship between Γ-spaces and connective spectra the reader may wish to consult Bousfield and Friedlander [10]. Let Γ^{op} denote the category of finite pointed sets. For n a non-negative integer let n^+ denote the pointed set $\{0, 1, \dots, n\}$ with 0 the base point. A Γ-object in a pointed category $\mathcal{C}$, with an initial and final object o, is a functor $F : \Gamma^{op} \to \mathcal{C}$ such that $F(0^+) = o$. Let cat_o denote the category of small pointed categories. Denote the category of simplicial sets (resp. pointed simplicial sets) by $\mathsf{s.sets}$ (resp. $\mathsf{s.sets}_o$). We shall make a minor abuse of terminology and refer to categories with enrichments in $\mathsf{s.sets}$ as simplicial categories. Let $\mathsf{cat}_o^{\mathsf{top}}$ (resp. $\mathsf{cat}_o^{\mathsf{s.sets}}$) denote the categories of small pointed topological (resp. simplicial) categories. The singular complex functor $\mathrm{Sin} : \mathsf{top} \to \mathsf{s.sets}$ prolongs to a functor $\mathrm{Sin} : \mathsf{cat}_o^{\mathsf{top}} \to \mathsf{cat}_o^{\mathsf{s.sets}}$. In particular if $\mathcal{C}$ is a Γ-topological category $\mathrm{Sin}\,\mathcal{C}$ is a Γ-simplicial category.

The passage from Γ-$\mathsf{cat}_o^{\mathsf{top}}$ to Γ-top_o is achieved with the aid of a *good* classfying space functor. If $\mathcal{C}$ is a topological category its nerve $\mathcal{C}_\bullet$ is a simplicial space, and the usual classifying space of $\mathcal{C}$ is the coend

$$\mathcal{C}_\bullet \otimes \Delta^\bullet \leftarrow \coprod_{[p]} \mathcal{C}_p \times \Delta^p \leftleftarrows \coprod_{[p]\to[q]} \mathcal{C}_q \times \Delta^p.$$

As pointed out by Segal [42] the usual classify space functor only has good properties for well-pointed categories, i.e., category in which the inclusion the map $\{id_X\} \to \mathcal{C}(X, X)$ is a cofibration for each object $X \in \mathcal{C}$. Most of Segal's op.cit. alternative realizations for a simplicial space X involve a functoral factorization of the map $\emptyset \to X$ into a cofibration $\emptyset \to X'$ and a weak equivalence $\phi : X' \to X$, and then defining $|X|' = |X'|$. Since the geometric realization $|\,\mathrm{Sin}\,X_\bullet| \otimes \Delta^\bullet$ of a simplicial space X is one such factorization, it follows that

$$\mathrm{B}\mathcal{C} := |\,\mathrm{Sin}\,\mathcal{C}_\bullet| \otimes \Delta^\bullet \tag{1.8.1}$$

is a good classifying space functor.

Let $\mathsf{Proj}(A)^{\mathsf{top}}$ denote the topological Γ-category of finitely generated A-modules, and let $K^{\mathrm{top}}(A)$ denote the associated connective spectrum. Denote the classifying space of the topological group $\mathrm{GL}^{\mathrm{top}} A$ by $\mathrm{BGL}^{\mathrm{top}} A$.

1.9 Lemma. *There is a natural isomorphism*

$$\pi_n K^{\mathrm{top}}(A) \simeq \pi_n(K_0(A) \times \mathrm{BGL}^{\mathrm{top}} A) \quad \textit{for } n \geq 0.$$

Proof. Applying Segal's Γ-space construction to $\mathsf{Proj}(A)^{\mathit{top}}$ we obtain a connective spectrum $K^{\mathrm{top}}(A)$ associated to the infinite loop space $\Omega\mathrm{BM}$, where

$$\mathrm{M} = \coprod_{P\in\mathsf{Proj}(A)} \mathrm{Aut}(P)^{\mathrm{top}}. \tag{1.9.1}$$

In particular

$$K_n^{\mathrm{top}}(A) = \pi_n\Omega\mathrm{BM} \cong \pi_n\big(K_0(A)\times \mathrm{BGL}^{\mathrm{top}}A\big)$$

for $n \geq 0$ (see [1]). □

1.10. Homotopy algebraic K-Theory

For X an admissible space and $A = \mathbf{C}(X)$ we have seen that there are natural isomorphisms $\boldsymbol{ku}^0(X) = K_0^{\mathrm{top}}(A) = K_0(A)$. The higher homotopy groups of the spectrum $K(A)$ are not necessarily isomorphic to the topological K-theory of X. If A is a stable C^*-algebra, however, then a conjecture of Karoubi, which was proved by Suslin and Wodzicki [44], states that the topological and algebraic K-theories of A are same. In order to reveal the connection between the topological K-theory of X and the algebraic K-theory of A, we must encode the topological structure of A combinatorially into the category $\mathsf{Proj}(A)$ with the simplicial ring $A(\Delta^\bullet)$, where $\Delta^n \subset \mathbf{R}^{n+1}$ is the geometric n-simplex and $A(\Delta^n)$ is the ring of A-valued continuous functions on Δ^n.

Technically $\mathsf{Proj}\big(A(\Delta^\bullet)\big)$ is a lax-functor, and $K\big(A(\Delta^\bullet)\big)$ is not a simplicial spectrum. One may rectify this defect as in [20], however, we shall use a different method. Clearly there is a canonical isomorphism $A(\Delta^n) = \mathbf{C}(\Delta^n \times X)$, and hence $\mathsf{Proj}\big(A(\Delta^n)\big)$ is equivalent to the category of complex vector bundles on $\Delta^n\times X$. Since all complex vector bundles on $\Delta^n\times X$ are isomorphic to vector bundles of the form $\Delta^n\times E$, where E is a vector bundle on X, the category $\mathsf{Proj}\big(A(\Delta^n)\big)$ is equivalent to the category $\mathrm{Sin}_n\,\mathsf{Proj}(A)^{\mathit{top}}$, whose objects are finitely generated projective A-modules and whose maps are families of A-linear maps parameterized by Δ^n (cf. [29]). Thus we obtain a simplicial category $\mathrm{Sin}_\bullet\mathsf{Proj}(A)^{\mathit{top}}$. Let

$$KH^{cont}(A) := \big|K\big(\mathrm{Sin}_\bullet\mathsf{Proj}(A)^{\mathit{top}}\big)\big|$$

and

$$KH_n^{cont}(A) := \pi_n KH^{cont}(A).$$

1.11 Remark. For an associative ring A, Weibel [52] defines a spectrum $KH(A)$ from the simplicial ring

$$\Delta^p A = A[t_0, \ldots, t_p] / \sum t_i - 1,$$

and shows that $KH(A)$ has excision and satisfies Zariski descent.

1.12 Theorem. *There is a natural weak equivalence*

$$KH^{cont}(A) \simeq K^{\mathrm{top}}(A),$$

and a right half-plane multiplicative spectral sequence

$$E^1_{p,q} = K_q(A(\Delta^p)) \Rightarrow K^{\mathrm{top}}_{p+q}(A)$$

whose edge map $K_p(A) \to K^{\mathrm{top}}_p(A)$ is the standard map between the algebraic and topological K-theories of A.

Proof. In the proof of the previous lemma we saw that $K^{\mathrm{top}}(A)$ is the connective spectrum associated to the infinite loop space $\Omega \mathrm{B} M$, where B is the good classifying space functor in (1.8.1) and M is the topological monoid in (1.9.1). Let $M(\Delta^p) = \coprod_{P \in \mathsf{Proj}(A)} \mathrm{Sin}_p \operatorname{Aut}(P)^{\mathrm{top}}$, and consider $M(\Delta^p)$ as a topological monoid with the discrete topology. Since Segal's Γ construction and Waldhausen's $S.$ construction give equivalent connective spectrum for categories with cofibrations and weak equivalences in which all cofibrations are split (see [51, §1.8]), it follows that the connective spectrum associated to $\Omega \mathrm{B} M(\Delta^p)$ is $K(A(\Delta^p))$. Applying the Fubini theorem [32, IX §8] for iterated coend and noting that $\mathrm{B} M(\Delta^p)$ is connected, we obtain a natural weak equivalence

$$\Omega \mathrm{B} M \simeq |\Omega \mathrm{B} M(\Delta^\bullet)|.$$

Moreover the deloopings of $\Omega \mathrm{B} M$ and $\Omega \mathrm{B} M(\Delta^\bullet)$ are compatible in the sense that n^{th} space of the spectrum associated to $\Omega \mathrm{B} M$ is naturally isomorphic to geometric realization of the n^{th} simplicial space of the simplicial spectrum associated to $\Omega \mathrm{B} M(\Delta^\bullet)$. Thus there is a natural weak equivalence $K^{\mathrm{top}}(A) \simeq KH^{cont}(A)$. The spectral sequence follows from the same arguments used in [52, 1.3].

The proof of the theorem is now straightforward save the multiplicative structure. Due to the lack of multiplicative structures of spectral sequences in the literature we give a separate discussion in an appendix. □

For an admissible space X, let X_+ denote the pointed space obtained from X by adding a disjoint base point. Let $F(X_+, \boldsymbol{ku})$ denote the

mapping spectrum and $F(X, \boldsymbol{ku})$ the cofiber of the natural map $\boldsymbol{ku} \cong F(T, \boldsymbol{ku}) \to F(X_+, \boldsymbol{ku})$, where T is a terminal spectrum. Since T is also initial we have

$$F(X_+, \boldsymbol{ku}) \cong \boldsymbol{ku} \vee F(X, \boldsymbol{ku}).$$

In particular $\pi_n F(X, \boldsymbol{ku}) \cong \boldsymbol{ku}^{-n}(X)$. Let $K(X) = K\big(\mathbf{C}(X)\big)$ denote the algebraic K-theory constructed from the exact category of complex vector bundles on X. The proof of the following is immediate from the preceding discussion.

1.13 Corollary. *$KH^{cont}(X)$ is the -1-connective cover of the mapping spectrum $F(X, \boldsymbol{ku})$, and there is a natural map $K(X) \to KH^{cont}(X)$ inducing a natural transformation*

$$KH^{cont}_*(X) \to \boldsymbol{ku}^{-*}(X)$$

of $\mathbb{Z}$-graded abelian group valued functors which is an isomorphism in non-negative homological degrees.

2. Localization and Excision

In this section we define for a pair (X, A), where X is an admissible space and $A \subseteq X$ is closed, a relative spectrum $KH^{cont}(X, A)$. Our formulation of $KH^{cont}(X, A)$ is a simplified version of a construction due to Thomason and Trobaugh [48].

2.1. Simplicial families of vector bundles

Let $\mathcal{C}$ denote the sheaf of continuous complex function on X. We embed the category of complex vector bundles on X in the abelian category of $\mathcal{C}$-modules via the local section functor $E \mapsto \Gamma(E)$. Let $\mathcal{E}(X)$ denote the category of bounded cochain complexes of vector bundles on X. The category $\mathcal{E}(X)$ is a category with cofibrations and weak equivalences, where the former are degree-wise split monomorphisms and the latter are quasi-isomorphisms. Applying Waldhausen's $S.$-construction to $\mathcal{E}(X)$ we obtain a spectrum that is weakly equivalent to $K(X)$ (see [19, 6.2] and [48, 1.11.7]). Let $\mathcal{E}_X(\Delta^n)$ denote the category whose objects are bounded cochain complexes of vector bundles on X and whose maps are Δ^n-families of cochain maps. Note that $\mathcal{E}_X(\Delta^\bullet)$ is a simplicial category, and it follows that we have a natural weak equivalence $KH^{cont}(X) \simeq |K(\mathcal{E}_X(\Delta^\bullet))|$.

For a pair (X, A) of admissible spaces let $\mathcal{E}(X, A)$ denote the full subcategory of bounded cochain complexes of vector bundles on X

which are acyclic on A, and let $\mathcal{E}_{(X,A)}(\Delta^n)$ denote the analogous full subcategory of $\mathcal{E}_X(\Delta^n)$. These subcategories inherit the structure of categories with cofibrations and weak equivalences. Let $K(X,A)$ be the K-theory spectrum associated to $\mathcal{E}(X,A)$, and $KH^{cont}(X,A)$ the geometric realization of the simplicial spectrum $K\big(\mathcal{E}_{(X,A)}(\Delta^\bullet)\big)$. The KH^{cont}-theories of X and A are related by

2.2 Theorem. *Given a pair (X,A) as above, there is a long exact sequence*

$$\begin{aligned}\cdots \to KH_n^{cont}(X) \to KH_n^{cont}(A) \to KH_{n-1}^{cont}(X,A) \to \cdots \\ \cdots \to KH_1^{cont}(X,A) \to KH_0^{cont}(X) \to KH_0^{cont}(A).\end{aligned}$$

Proof. The category $\mathcal{E}_X(\Delta^n)$ satisfies the extension and saturation axioms, and it has a cylinder functor satisfying the cylinder axiom (see [51, 48]). Let $v\mathcal{E}_X(\Delta^n)$ denote the class of maps in $\mathcal{E}_X(\Delta^n)$ which are quasi-isomorphism over A. The subcategory $v\mathcal{E}_X(\Delta^n)$ also satisfies the extension and saturation axioms. Applying Waldhausen's fibration theorem [51, 1.6.4] we obtain a homotopy fibration

$$K\big(\mathcal{E}_{(X,A)}(\Delta^n)\big) \to K\big(\mathcal{E}_X(\Delta^n)\big) \to K\big(v\mathcal{E}_X(\Delta^n)\big).$$

By the well known fact that the geometric realization functor of spectra preserves homotopy fibration sequences we get a homotopy fibration

$$KH^{cont}(X,A) \to KH^{cont}(X) \to |K\big(v\mathcal{E}_X(\Delta^\bullet)\big)|.$$

Since the restriction map $\mathcal{E}_X(\Delta^n) \to \mathcal{E}_A(\Delta^n)$ factors through $v\mathcal{E}_X(\Delta^n)$ we must show that

$$\pi_n|K\big(v\mathcal{E}_X(\Delta^\bullet)\big)| \cong \begin{cases}\pi_n KH^{cont}(A) & \text{for } n>0\\ \operatorname{im}\big(\boldsymbol{ku}^0(X) \to \boldsymbol{ku}^0(A)\big) & \text{for } n=0.\end{cases}$$

Let $\mathcal{E}_A(\Delta^n)^v$ be the full subcategory of $\mathcal{E}_A(\Delta^n)$ consisting of bounded complexes $\{E^i, d\}$ of vector bundles over $\Delta^n \times A$ such that $\sum(-1)^i[E^i]$ lies in the image of the restriction map $\boldsymbol{ku}^0(X) \to \boldsymbol{ku}^0(A)$. By the Thomason-Trobaugh [48] cofinality Theorem 1.10.1

$$\pi_n|K\big(E_A(\Delta^\bullet)^v\big)| \cong \begin{cases}\pi_n KH^{cont}(A) & \text{for } n>0\\ \operatorname{im}\big(\boldsymbol{ku}^0(X) \to \boldsymbol{ku}^0(A)\big) & \text{for } n=0.\end{cases}$$

The proof will follow from [48, 5.2] by showing that a bounded complex F on $\Delta^n \times A$ is isomorphic to the restriction of a bounded complex on $\Delta^n \times X$ if and only if the class $[F] \in K_0(\Delta^n \times A)$ is in the image of the

restriction map $K_0(\Delta^n \times X) \to K_0(\Delta^n \times A)$ and any map $f : F \to G$ on $\Delta^n \times A$ prolongs to a map on $\Delta^n \times X$ (cf.[54, §3]).

Atiyah [3, II §2.6] shows that for any compact space Y the map χ that sends a bounded complex C of vector bundles on Y to the element $\sum(-1)^i[C^i] \in \boldsymbol{ku}^0(Y)$ is an isomorphism from the set of homotopy classes of bounded complexes to $\boldsymbol{ku}^0(Y)$, where a homotopy between E_0 and E_1 consists of a bounded complex E of vector bundles on $[0,1] \times E$ and isomorphisms $E_i \cong E|_{\{i\}\times X}$ for $i = 0, 1$. Since his proof extends to admissible spaces, it follows that $\boldsymbol{ku}^0(A) \cong \boldsymbol{ku}^0(\Delta^n \times A)$ is isomorphic to the set of homotopy classes of bounded complex vector bundles on A. Similarly the relative group $\boldsymbol{ku}^0(\Delta^n \times X, \Delta^n \times A)$ is isomorphic to the quotient, by the previous homotopy relation, of finite complexes on $\Delta^n \times X$ which are acyclic on $\Delta^n \times A$. From the exact sequence

$$\boldsymbol{ku}^0(\Delta^n \times X, \Delta^n \times A) \to \boldsymbol{ku}^0(X) \to \boldsymbol{ku}^0(A)$$

we see that any bounded complex F of vector bundles on $\Delta^n \times A$ is homotopic to the restriction of a bounded complex E of vector bundles on $\Delta^n \times X$ if and only if

$$\chi(F) = \sum(-1)^i[F^i] \in \mathrm{im}(\boldsymbol{ku}^0(X) \to \boldsymbol{ku}^0(A)).$$

By the general theory of vector bundles over paracompact spaces we may replace homotopic by isomorphic. Let E and F be bounded complexes of vector bundles on $\Delta^n \times X$, and let $\mathrm{Hom}(E, F)$ be the difference kernel subsheaf of the obvious vector bundle maps

$$\prod_n \mathrm{Hom}(E^n, F^n) \rightrightarrows \prod_n \mathrm{Hom}(E^{n-1}, F^n).$$

Let $j : \Delta^n \times A \to \Delta^n \times X$ denote the inclusion. Since the sheaf $\mathrm{Hom}(E, F)$ is fine and any map $\beta : j^*E \to j^*F$ of bounded complexes of vector bundles corresponds to a section over A, there exist a map $\alpha : E \to F$ such that $j^*\alpha = \beta$. Thus any map defined over $\Delta^n \times A$ can be prolonged to a map over $\Delta^n \times X$. $\square$

2.3. Excision

If X is a compact space, then $\mathbf{C}(X)$ and the ideal $\mathfrak{a} = \{f \in \mathbf{C}(X) : f|_A = 0\}$ are C^*-algebras. Corollary 10.4 in [44] asserts that the algebraic K-theory of $\mathfrak{a}$ does not depend on the inclusion $\mathfrak{a} \to \mathbf{C}(X)$ as a two-sided ideal. Applying this result to $KH^{cont}(X, A)$ we obtain a weak equivalence $KH^{cont}(X, A) \to KH^{cont}(X - U, A - U)$, where U is open subset of X whose closure is contained in A. It is also

noted op.cit. Remark 1.13 that excision holds for locally multiplicatively convex Fréchet algebras with uniformly bounded approximate units. If X is admissible then $\mathbf{C}(X)$ and $\mathfrak{a}$ are locally multiplicatively convex Fréchet algebras, but I do not know if $\mathfrak{a}$ has uniformly bounded approximate units. Thus to prove excision for admissible pairs (X, A) we compare $\pi_n KH^{cont}(X, A)$ to $\boldsymbol{ku}^{-n}(X, A)$.

Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A/A \arrow[r] & X/A \end{tikzcd}$$

where f is the quotient map collapsing A to a point. There is an induced map $KH^{cont}(X/A, A/A) \to KH^{cont}(X, A)$. Since A/A is a point and $A/A \hookrightarrow X/A$ is a retract, the long exact sequence of $(X/A, A/A)$ splits into a collection of split exact sequences

$$0 \to \pi_n KH^{cont}(X/A, A/A) \to \boldsymbol{ku}^{-n}(X/A) \to \boldsymbol{ku}^{-n}(A/A) \to 0.$$

Thus $\pi_n KH^{cont}(X/A, A/A) \cong \boldsymbol{ku}^{-n}(X, A)$ for $n \geq 0$. By a standard application of the five lemma we obtain

2.4 Proposition. *If U is an open subset of X whose closure is contained in a closed subset A, then $(X - U, A - U) \to (X, A)$ induces a weak equivalence $KH^{cont}(X, A) \to KH^{cont}(X - U, A - U)$. Moreover $KH^{cont}_n(X, A) \cong \boldsymbol{ku}^{-n}(X, A)$ for $n \geq 0$.*

2.5. Bass K-Theory

To recover the positive topological K-theory of X we need to prolong the algebraic K-theory spectrum $K(X)$ to the non-connective spectrum $K^B(X)$ as in [52, 48], where $\pi_n K^B(X)$, for $n < 0$, is the Bass K-theory of $\mathbf{C}(X)$. To construct $K^B(X)$ we follow Thomason and Trobaugh [48, §6] and inductively define a natural sequence F^{-k} of spectra for $k \geq 0$.

Let $X^{alg} = \operatorname{Spec} \mathbf{C}(X)$, and let $X^{alg}[t]$ and $X^{alg}[t, t^{-1}]$ denote the affine schemes associated to the ring of polynomials and Laurent polynomials with coefficients in $\mathbf{C}(X)$ respectively. We set

$$F^0(X) := K(X).$$

For $k < 0$, let $F^{k+1}(X^{alg}[t]) \vee^h_{F^{k+1}(X)} F^{k+1}(X^{alg}[t^{-1}])$ denote the homotopy push-out of

$$\begin{tikzcd} F^{k+1}(X) \arrow[r] \arrow[d] & F^{k+1}(X^{alg}[t]) \\ F^{k+1}(X^{alg}[t^{-1}]), & \end{tikzcd}$$

and let $G^k(X)$ denote the homotopy cofiber of the map

$$F^{k+1}(X^{alg}[t]) \bigvee_{F^{k+1}(X)}^{h} F^{k+1}(X^{alg}[t^{-1}]) \xrightarrow{d} F^{k+1}(X^{alg}[t,t^{-1}]). \tag{2.5.1}$$

Finally we set

$$F^k(X) := \Omega G^k(X) = F(S^1, G^k(X)).$$

There are natural maps $F^k(X) \to \Omega F^k(X^{alg}[t,t^{-1}]) \xrightarrow{\Omega d} F^{k-1}(X)$, where the first map is adjoint to

$$F^k(X) \wedge S^1 \xrightarrow{p^* \wedge t} F^k(X^{alg}[t,t^{-1}]) \wedge K(X^{alg}[t,t^{-1}]) \to F^k(X^{alg}[t,t^{-1}]).$$

The spectrum $K^B(X)$ is the homotopy colimit

$$K^B(X) = \operatorname*{hocolim}_{k \leq 0} F^k(X).$$

Let $\boldsymbol{KU}$ denote the periodic topological K-theory spectrum. Recall that

$$\boldsymbol{KU} = \boldsymbol{ku}[\beta^{-1}] = \operatorname*{hocolim}_{n \geq 0} \Omega^{2n} \boldsymbol{ku},$$

where $\beta : \boldsymbol{ku} \to \Omega^2 \boldsymbol{ku}$ is multiplication by the Bott element.

2.6 Lemma. *There is a natural transformation of Spt valued functors*

$$K^B(X) \to F(X, \boldsymbol{KU}).$$

In particular there is a natural transformation of abelian group valued functors $K^B_n(X) \to \boldsymbol{KU}^{-n}(X)$.

Proof. To prove the lemma we shall need to examine the topological K-theory of the product $\mathbf{P}^1 \times X$. To simplify the discussion, we identify the complex projective line with the one point compactification $\mathbf{C} \cup \infty$ of the complex line. For any $\lambda \in \mathbf{P}^1$, let s_λ denote the section of the projection $p : \mathbf{P}^1 \times X$ given by $s_\lambda(x) = (\lambda, x)$. Identifying X with the

closed subspace $\{\infty\} \times X$ via the section s_∞ and observing that the long exact sequence of the pair $(\mathbf{P}^1 \times X, X)$ is split by the projection p^*, we obtain a natural family of split exact sequences

$$0 \to \boldsymbol{ku}^n(\mathbf{P}^1 \times X, X) \to \boldsymbol{ku}^n(\mathbf{P}^1 \times X) \to \boldsymbol{ku}^n(X) \to 0. \tag{2.6.1}$$

The group $\boldsymbol{ku}^n(\mathbf{P}^1 \times X, X)$ is naturally isomorphic to $\boldsymbol{ku}^{n-2}(X)$. This can be seen as follows. Apply excision to the pair $(\mathbf{P}^1 \times X_+, \mathbf{P}^1 \vee X_+)$ to get natural isomorphisms

$$\boldsymbol{ku}^n(\mathbf{P}^1 \times X_+, \mathbf{P}^1 \vee X_+) \cong$$
$$\boldsymbol{ku}^n(\mathbf{P}^1 \times X_+ \setminus \mathbf{P}^1 \times \{o\}, \mathbf{P}^1 \vee X_+ \setminus \mathbf{P}^1 \times \{o\}) \cong \boldsymbol{ku}^n(\mathbf{P}^1 \times X, X).$$

Then use the natural isomorphisms

$$\begin{aligned} \boldsymbol{ku}^n(\mathbf{P}^1 \times X_+, \mathbf{P}^1 \vee X_+) &\cong \boldsymbol{ku}^n(\mathbf{P}^1 \wedge X_+, \emptyset_+) \\ &\cong \boldsymbol{ku}^n(\Sigma^2 X_+, \emptyset_+) \\ &\cong \boldsymbol{ku}^{n-2}(X_+, \emptyset_+) \\ &\cong \boldsymbol{ku}^{n-2}(X). \end{aligned}$$

It follows that we may rewrite the split exact sequence (2.6.1) as

$$0 \to \boldsymbol{ku}^{n-2}(X) \to \boldsymbol{ku}^n(\mathbf{P}^1 \times X) \to \boldsymbol{ku}^n(X) \to 0.$$

We now consider the Mayer-Vietoris sequence of the open cover $\{U_0, U_\infty\}$ of $\mathbf{P}^1 \times X$ given by the complements of the zero section and the section at infinity. There are homotopy equivalences $U_\infty \simeq X \simeq U_0$ and $S^1 \times X \simeq U_0 \cap U_\infty$. Using the previous step, with $\mathbf{P}^1$ replaced by the real projective line, the Mayer-Vietoris sequence take the form of the middle vertical sequence with each horizontal sequence split exact

and each vertical sequence exact.

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
\boldsymbol{ku}^{n-2}(X) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n-1}(U_0 \cap U_\infty) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n-1}(X) \arrow[d, "0"] \\
\boldsymbol{ku}^{n-2}(X) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n}(\mathbf{P}^1 \times X) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n}(X) \arrow[d, "\Delta"] \\
0 \arrow[r] \arrow[d] & \boldsymbol{ku}^{n}(U_0) \oplus \boldsymbol{ku}^{n}(U_\infty) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n}(X) \oplus \boldsymbol{ku}^{n}(X) \arrow[d] \\
\boldsymbol{ku}^{n-1}(X) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n}(U_0 \cap U_\infty) \arrow[r] \arrow[d] & \boldsymbol{ku}^{n}(X) \arrow[d, "0"] \\
{} & {} & {}
\end{tikzcd}$$

Rewriting the Mayer-Vietoris sequence in terms of function spectra we obtain, up to homotopy, the following cofiber sequence

$$F(U_0, \boldsymbol{ku}) \bigvee_{F(X,\boldsymbol{ku})}^{h} F(U_\infty, \boldsymbol{ku}) \to F(U_0 \cap U_\infty, \boldsymbol{ku}) \to F(X, \Omega \boldsymbol{ku}). \tag{2.6.2}$$

If $x \in X$ let $\mathfrak{m}_x$ denote the maximal ideal of continuous complex valued functions vanishing at x. Since paracompact spaces are normal the map $x \mapsto \mathfrak{m}_x$ from X to the set of maximal ideals of $\mathbf{C}(X)$ is injective, and induces a map $\phi : X \to X^{alg}$ of locally ringed spaces. We may extend the map ϕ to a commutative diagram

$$\begin{tikzcd}
U_0 \arrow[d] & U_0 \cap U_1 \arrow[l] \arrow[r] \arrow[d] & U_1 \arrow[d] \\
X^{alg}[t] & X^{alg}[t, t^{-1}] \arrow[l] \arrow[r] & X^{alg}[t^{-1}].
\end{tikzcd}$$

Thus we obtain

$$\begin{tikzcd}
K(X^{alg}[t]) \bigvee^h_{K(X)} K(X^{alg}[t^{-1}]) \arrow[r] \arrow[d] & F(U_0, \boldsymbol{ku}) \bigvee^h_{F(X,\boldsymbol{ku})} F(U_\infty, \boldsymbol{ku}) \arrow[d] \\
K(X^{alg}[t, t^{-1}]) \arrow[r] \arrow[d] & F(U_0 \cap U_\infty, \boldsymbol{ku}) \arrow[d] \\
G^{-1}(X) \arrow[r] & F(X, \Omega \boldsymbol{ku})
\end{tikzcd}$$

and a natural map $F^{-1}(X) \to F(X, \Omega^2 \boldsymbol{ku})$ such that

$$\begin{tikzcd} K(X) \arrow[r] \arrow[d] & F^{-1}(X) \arrow[d] \\ F(X, \boldsymbol{ku}) \arrow[r, "\beta"] & (X, \Omega^2 \boldsymbol{ku}) \end{tikzcd}$$

commutes, where β is the Bott map. By induction and the cofibration sequences (2.5.1) and (2.6.2) we obtain natural maps

$$K^B(X) \to \operatorname*{hocolim}_{n \geq 0} F(X, \Omega^{2n} \boldsymbol{ku}) \to F(X, \boldsymbol{KU}).$$

□

Let $K^B H^{cont}(X)$ denote the geometric realization of the simplicial spectrum

$$K^B(\Delta^\bullet \times X),$$

and set $K^B H_n^{cont}(X) := \pi_n K^B H^{cont}(X)$.

2.7 Theorem. *For any admissible space X, there is a natural isomorphism*

$$K^B H_n^{cont}(X) \cong \boldsymbol{ku}^{-n}(X).$$

Proof. First we shall show that $K^B H_*^{cont}$ is a graded generalized cohomology theory on the homotopy category of finite dimensional CW-complexes. For $A \subseteq X$ a closed subspace, let $K^B(\Delta^n \times X, \Delta^n \times A)$ be the homotopy limit of $K^B(\Delta^n \times X) \to K^B(\Delta^n \times A) \leftarrow T$, where T is a fixed terminal object in Spt. Thus $[n] \mapsto K^B(\Delta^n \times X, \Delta^n \times A)$ is a simplicial spectrum, and

$$K^B(\Delta^\bullet \times X, \Delta^\bullet \times A) \to K^B(\Delta^\bullet \times X) \to K^B(\Delta^\bullet \times A)$$

is a pointwise fibration. Let

$$K^B H^{cont}(X, A) = |K^B H^{cont}(\Delta^\bullet \times X, \Delta^\bullet \times A)|.$$

By construction there is a long exact sequence

$$\cdots \to K^B H_n^{cont}(X) \to K^B H_n^{cont}(A) \to K^B H_{n-1}^{cont}(X, A) \to \cdots$$

Let $H : \Delta^1 \times X \to Y$ be a homotopy between $f_0, f_1 : X \to Y$. The maps $(s_i, s_0 \cdots \hat{s}_i \cdots s_q) : \Delta^{q+1} \to \Delta^q \times \Delta^1$ for $0 \leq i \leq q$ determine a triangulation of $\Delta^q \times \Delta^1$ and a simplicial homotopy $\Delta[1] \times K^B(\Delta^\bullet \times Y) \to K^B(\Delta^\bullet \times X)$ from f_0^* to f_1^*. Since the geometric realization functor preserves both simplicial and topological homotopies [15,

Ch. X Prop. 2.1], it follows that $K^B H_*^{cont}(X)$ and $K^B H_*^{cont}(X, A)$ are homotopy invariant functors.

If $X = \coprod X_\alpha$, then $\mathbf{C}(X) = \prod \mathbf{C}(X_\alpha)$. Since good classifying spaces functors preserve products, $KH^{cont}(X) \cong \prod KH^{cont}(X_\alpha)$, and by [48, 6.7] this implies that $K^B H^{cont}(X) \cong \prod K^B H^{cont}(X_\alpha)$. Thus the functor $X \mapsto K^B H^{cont}(X)$ is strongly additive.

We now turn our attention to proving excision. Let

$$(\Delta^n \times (X, A)) := (\Delta^n \times X, \Delta^n \times A).$$

Recall from the proof of 2.2 that there is a distinguished triangle

$$K(\Delta^n \times (X, A)) \to K(\Delta^n \times X) \to K(\Delta^n \times A)^{\sim}$$

where $K(\Delta^n \times A)^{\sim}$ is the spectrum $K(E_A(\Delta^n)^v)$.

Consider the following commutative diagram in the homotopy category of spectra

$$\begin{tikzcd}
K(\Delta^n \times (X, A)) \arrow[d] & K^B(\Delta^n \times (X, A)) \arrow[r, dotted] \arrow[d] & \Box \arrow[d, dotted] \\
K(\Delta^n \times X) \arrow[r] \arrow[d] & K^B(\Delta^n \times X) \arrow[r] \arrow[d] & P_{-1}K^B(\Delta^n \times X) \arrow[d, dotted] \\
K(\Delta^n \times A)^{\sim} \arrow[r] & K^B(\Delta^n \times A) \arrow[r] & P_{-1}K^B(\Delta^n \times A)^{\sim}
\end{tikzcd}$$

where the horizontal and vertical maps are distinguished triangles. The spectrum $P_{-1}K^B(\Delta^n \times X)$ is the -1-Postnikov section of $K^B(\Delta^n \times X)$ and

$$\pi_q P_{-1} K^B(\Delta^n \times A)^{\sim} = \begin{cases} 0 & q > 0 \\ \operatorname{coker}\big(K_0(\Delta^n \times X) \to K_0(\Delta^n \times A)\big) & q = 0 \\ K_q^B(\Delta^n \times X) & q < 0. \end{cases} \tag{2.7.1}$$

By the 3×3-lemma for triangulated categories [5, Prop. 1.1.11] we may replace the square and the dotted maps in the preceding diagram by a spectrum E and maps to obtain a commutative diagram, up to sign, in the homotopy category of spectra in which the rows and columns are distinguished triangles. From (2.7.1) it follows that E is the -1-Postnikov section of $K^B(\Delta^n \times X, \Delta^n \times A)$.

Given $U \subseteq A \subseteq X$ with U open and A closed, set $\mathfrak{a}(\Delta^n) = \{f \in \mathbf{C}(\Delta^n \times X) : f|_{\Delta^n \times A} = 0\}$. The set $A - U$ is closed in $X - U$, and the following sequences of rings

$$0 \to \mathfrak{a}(\Delta^n) \to \mathbf{C}(\Delta^n \times X) \to \mathbf{C}(\Delta^n \times A) \to 0$$
$$0 \to \mathfrak{a}(\Delta^n) \to \mathbf{C}\big(\Delta^n \times (X - U)\big) \to \mathbf{C}\big(\Delta^n \times (A - U)\big) \to 0$$

are exact by the Tietze extension theorem. Using Proposition 1.4 we see that topological excision holds if algebraic excision holds. Now a theorem of Bass [4, Thm. XII. 8.3] asserts that non-positive algebraic K-theory of ring has excision. In particular $\pi_p E = K_p^B(\mathfrak{a}(\Delta^n)$ for $p < 0$, whence

$$\pi_n |K^B(\Delta^\bullet \times (X, A))| \cong \pi_n |K^B(\Delta^\bullet \times (X - U, A - U))|$$

for $n < 0$. Since topological excision holds for KH^{cont} by Corollary 2.4 and

$$KH^{cont}(X, A) \to K^B H^{cont}(X, A) \to |P_{-1} K^B(\Delta^\bullet \times (X, A))|$$

is an distinguished triangle, topological excision holds for $K^B H^{cont}$.

By Brown's representation theorem we know that there is a spectrum F that represents $K^B H^{cont}_*$ on the homotopy category finite CW-complexes. Thus there there is a natural isomorphism $F^{-n}(X) \cong K^B H^{cont}_n(X)$ for all integers n, and each natural transformation

$$K^B H^{cont}_n(X) \to G^{-n}(X)$$

of generalized cohomology theories corresponds to a map $F \to G$ of spectra. By Lemma 2.6 and Corollary 1.13 there exists a map $f : F \to \boldsymbol{KU}$, which factors through $\boldsymbol{ku}$ and induces an isomorphism $K^B H^{cont}_* \to \boldsymbol{ku}^{-*}$. Thus $F = \boldsymbol{ku}$ proving the theorem. $\square$

3. Hypercohomology

We shall assume that the reader is familiar with the terminology of (stable) closed model categories. Those readers unfamiliar with such objects may wish to consult A.1.

3.1. Presheaves of spectra

Let X be a topological space. We denote by $\mathsf{Spt}(X)$ the category of presheaves of spectra on X. Jardine [27] has proved that $\mathsf{Spt}(X)$ has a proper simplicial closed model structure. A map $\varphi : \mathcal{E} \to \mathcal{F}$ in $\mathsf{Spt}(X)$ is called (i) a weak equivalence if the induced map $\tilde{\pi}_* \varphi$ of sheaves of graded abelian groups is an isomorphism; (ii) a cofibration if for each open subset $U \subseteq X$ the map $\varphi(U)$ is a cofibration of spectra and (iii) a fibration if φ has the right lifting property with respect to the class of maps that are both weak equivalences and cofibrations.

3.2 Remark. Jardine actually proved that the category $\mathsf{Spt}(\mathbf{C})$ of presheaves of spectra on an arbitrary Grothendieck site $\mathbf{C}$ carries the structure of a proper simplicial closed model category.

3.3 Proposition. *$\mathsf{Spt}(X)$ is a stable closed model category.*

Proof. We must show that $\Sigma, \Omega : \mathsf{Ho}\,\mathsf{Spt}(X) \to \mathsf{Ho}\,\mathsf{Spt}(X)$ are adjoint equivalences (see A.1). Since the functors $S^1 \wedge -$ and $(\ )_o^{S^1}$ are an adjoint pair, it is enough (cf. [14, Theorem 9.7]) to show that for each cofibrant presheaf $\mathcal{A}$ and each fibrant presheaf $\mathcal{E}$ a map $\varphi : \mathcal{A} \to \mathcal{E}_o^{S^1}$ is a weak equivalence if and only if its adjoint $\varphi^\flat : S^1 \wedge \mathcal{A} \to \mathcal{E}$ is a weak equivalence. Since there are natural isomorphisms $\tilde{\pi}_*(S^1 \wedge \mathcal{A}) \cong \tilde{\pi}_{*-1}\mathcal{A}$ and $\tilde{\pi}_*\mathcal{E}_o^{S^1} \cong \tilde{\pi}_{*+1}\mathcal{E}$, it follows that $\tilde{\pi}_*\varphi$ is an isomorphism if and only if $\tilde{\pi}_*\varphi^\flat$ is an isomorphism. □

3.4. Derived functors

Let $f : X \to Y$ be a continuous map. Associated to f are the adjoint functors

$$\begin{tikzcd} \mathsf{Spt}(Y) \arrow[r, shift left, "f^{\#}"] & \mathsf{Spt}(X) \arrow[l, shift left, "f_{\#}"] \end{tikzcd}$$

where $f_{\#}\mathcal{E}(V) = \mathcal{E}(f^{-1}V)$ and $f^{\#}\mathcal{F}(U) = \operatorname{colim}_{V \supseteq f(U)} \mathcal{F}(V)$. The functor $f^{\#}$ preserves cofibrations and weak equivalences, and this is enough to ensure that the total derived functors

$$\begin{tikzcd} \mathsf{Ho}\,\mathsf{Spt}(Y) \arrow[r, shift left, "\mathsf{L}f^{\#}"] & \mathsf{Ho}\,\mathsf{Spt}(X) \arrow[l, shift left, "\mathsf{R}f_{\#}"] \end{tikzcd}$$

exist and form an adjoint pair. Since the functor $f^{\#}$ preserves weak equivalences, $\mathsf{L}f^{\#}$ is essentially the functor $f^{\#}$, and there is a commutative diagram

$$\begin{tikzcd} \mathsf{Spt}(Y) \arrow[r, "f^{\#}"] \arrow[d, "\gamma_Y"'] & \mathsf{Spt}(X) \arrow[d, "\gamma_X"] \\ \mathsf{Ho}\,\mathsf{Spt}(Y) \arrow[r, "\mathsf{L}f^{\#}"] & \mathsf{Ho}\,\mathsf{Spt}(X). \end{tikzcd}$$

Let $f : X \to o$ be the map to a point. Given $\mathcal{E} \in \mathsf{Spt}(X)$ the *hypercohomology* of X with coefficients in $\mathcal{E}$ is the graded abelian group

$$\mathbb{H}^*(X; \mathcal{E}) := \pi_{-*}\mathsf{R}f_{\#}\mathcal{E}.$$

3.5. Postnikov towers

Let

$$\mathcal{E}_N \xleftarrow{\varphi_1} \mathcal{E}_{N+1} \xleftarrow{\varphi_2} \mathcal{E}_{N+2} \xleftarrow{\varphi_3} \cdots$$

be a tower of maps in $\mathsf{Ho}\,\mathsf{Spt}(X)$. The *homotopy limit* of this tower is an object $\mathcal{E}_\infty$ together with a distinguished triangle

$$\mathcal{E}_\infty \to \prod \mathcal{E}_i \xrightarrow{1-\varphi} \prod \mathcal{E}_i \to \Sigma\mathcal{E}_\infty.$$

It readily follows that the total right derived functor $\mathsf{R}f_\#$ preserves homotopy limits. Note that $\mathcal{E}_\infty$ is uniquely defined up to isomorphism in $\mathsf{Ho}\,\mathsf{Spt}(X)$, and for each open subset $U \subseteq X$ there is a short exact sequence

$$0 \to \lim{}^1 \pi_{*+1}\mathcal{E}_i(U) \to \pi_*\mathcal{E}_\infty(U) \to \lim \pi_*\mathcal{E}_i(U) \to 0. \qquad (3.5.1)$$

Since the stalk of $\pi_*\mathcal{E}_\infty$ at $x \in X$ is the colimit of the filtered posets of open neighborhoods of x and (3.5.1) is a sequence of graded abelian groups, it follows that there is an analogous short exact sequence on stalks.

Let $\mathcal{E} \in \mathsf{Spt}(X)$ be a presheaf and assume that there exists an integer N such that for every $U \subseteq X$ and every $i < N$ we have $\pi_i\mathcal{E}(U) = 0$. Associated to $\mathcal{E}$ is a Postnikov tower

$$o \leftarrow P_N\mathcal{E} \leftarrow P_{N+1}\mathcal{E} \leftarrow P_{N+2}\mathcal{E} \leftarrow \cdots$$

with homotopy fiber squares

$$\begin{array}{ccc} K(\pi_t\mathcal{E}, t) & \longrightarrow & P_t\mathcal{E} \\ \downarrow & & \downarrow \\ o & \longrightarrow & P_{t-1}\mathcal{E}, \end{array}$$

where $K(\pi_t\mathcal{E}, t)$ is a presheaf of Eilenberg-Mac Lane spectrum of type $(\pi_t\mathcal{E}, t)$. In addition there is a compatible family of maps $\mathcal{E} \to P_t\mathcal{E} \in \mathsf{Spt}(X)$ inducing isomorphisms $\pi_i\mathcal{E} \cong \pi_i P_t\mathcal{E}$ for $i \leq t$. For $i > t$ we have $\pi_i P_t\mathcal{E} = 0$, and there is a natural weak equivalence $\mathcal{E} \to \lim_t P_t\mathcal{E}$. In particular $\mathcal{E}$ represents the homotopy limit of its Postnikov tower.

Let $\mathsf{s.Spt}(X)$ denote the category of simplicial objects in $\mathsf{Spt}(X)$. A theorem of Reedy [40] (cf. A.5) asserts that $\mathsf{s.Spt}(X)$ is also a proper simplicial closed model category. The pair $(f^\#, f_\#)$ prolongs pointwise to a pair $(\mathsf{s}.f^\#, \mathsf{s}.f_\#)$ of simplicial functors. Since $f_\#$ has a left adjoint

it preserves limits, and it follows that the prolonged functors form an adjoint pair. The total derived functors

$$\mathsf{s.Spt}(Y) \underset{R(\mathsf{s}.f_{\#})}{\overset{L(\mathsf{s}.f^{\#})}{\rightleftarrows}} \mathsf{s.Spt}(X)$$

exist and also form an adjoint pair.

Let $\mathcal{E}_\bullet$ a simplicial presheaf of spectra on X and assume that there exists an integer N such that for all $i < N$ and every open subset U of X the simplicial abelian group $\pi_i\mathcal{E}_\bullet(U)$ is trivial. By the canonical properties of the Postnikov tower [28, Ch.4] there is a tower

$$o \leftarrow P_N\mathcal{E}_\bullet \leftarrow P_{N+1}\mathcal{E}_\bullet \leftarrow P_{N+2}\mathcal{E}_\bullet \leftarrow \cdots$$

of simplicial presheaves on X. Let $|\ \ |_L : \mathsf{Ho\, s.Spt}(X) \to \mathsf{Ho\, Spt}(X)$ denote the total left derived functor of the realization functor $|\ \ |$, and consider the tower

$$o \leftarrow |P_N\mathcal{E}_\bullet|_L \leftarrow |P_{N+1}\mathcal{E}_\bullet|_L \leftarrow |P_{N+2}\mathcal{E}_\bullet|_L \leftarrow \cdots \tag{3.5.2}$$

in the homotopy category $\mathsf{Ho\, Spt}(X)$.

3.6 Lemma. *For each integer $t \geq N$ there is a distinguished triangle*

$$|K(\pi_t\mathcal{E}_\bullet, t)|_L \to |P_t\mathcal{E}_\bullet|_L \to |P_{t-1}\mathcal{E}_\bullet|_L \to \Sigma|K(\pi_t\mathcal{E}_\bullet, t)|_L$$

in $\mathsf{Ho\, Spt}(X)$.

Proof. First we shall show that for each t the pull-back square

$$\begin{tikzcd} F_t\mathcal{E}_\bullet \arrow[r, "k_\bullet"] \arrow[d] & P_t\mathcal{E}_\bullet \arrow[d, "p^t_\bullet"] \\ o \arrow[r] & P_{t-1}\mathcal{E}_\bullet \end{tikzcd}$$

in $\mathsf{s.Spt}(X)$ is a homotopy fiber square. Let $P_t\mathcal{E}_\bullet \xrightarrow{\iota} P'_t\mathcal{E}_\bullet \xrightarrow{q} P_{t-1}\mathcal{E}_\bullet$ be factorization of $p^t_\bullet$ with ι a trivial cofibration and q a fibration. The homotopy fiber of $p^t_\bullet$ is $F'_t\mathcal{E}_\bullet = 0 \times_{P_{t-1}\mathcal{E}_\bullet} P'_t\mathcal{E}_\bullet$. For each non-negative integer p and each open subset $U \subseteq X$ the maps $p^t_p(U)$ and $q_p(U)$ are fibrations and $\iota_p(U)$ is a trivial cofibration, thus $F_t\mathcal{E}_p \to F'_t\mathcal{E}_p$ is a weak equivalence. This proves that $F_t\mathcal{E}_\bullet \to F'_t\mathcal{E}_\bullet$ is a weak equivalence, and it follows that $F_t\mathcal{E}_\bullet$ is the homotopy fiber of $p^t_\bullet$.

Next we shall show that $P_{t-1}\mathcal{E}_\bullet$ is the homotopy cofiber of $k_\bullet$. Let $F_t\mathcal{E}_\bullet \xrightarrow{\epsilon} F''_t\mathcal{E}_\bullet \xrightarrow{\theta} P_t\mathcal{E}_\bullet$ be a factorization of $k_\bullet$ with ϵ a cofibration and θ a trivial fibration. We must show that the natural map

$$o \coprod_{F_t\mathcal{E}_\bullet} F''_t\mathcal{E}_\bullet \to P_{t-1}\mathcal{E}_\bullet \tag{3.6.1}$$

is a weak equivalence. For each non-negative integer p and each open subset $U \subseteq X$ the map $\epsilon_p(U)$ is a cofibration and $\theta_p(U)$ is a trivial fibration. Since $o \bigsqcup_{F_t\mathcal{E}_p(U)} F_t''\mathcal{E}_p(U)$ is the homotopy cofiber of $k_p(U)$, and $\mathsf{Spt}(X)$ is a stable closed model category, it follows that map in (3.6.1) is a weak equivalence.

The proof follows readily from the fact that $| \ \ |_{\mathsf{L}}$ preserves homotopy cofibration sequences [25, Proposition 6.4.1] and that there is a natural weak equivalence $F_t\mathcal{E}_\bullet \to K(\pi_t\mathcal{E}_\bullet, t)$. $\square$

Let $\operatorname{hlim} |P_t\mathcal{E}_\bullet|_{\mathsf{L}}$ denote the homotopy limit in $\mathsf{Ho\,Spt}(X)$ of the sequence (3.5.2). There is a natural map $|\mathcal{E}_\bullet|_{\mathsf{L}} \to \operatorname{hlim} |P_t\mathcal{E}_\bullet|_{\mathsf{L}}$. A proof of the following is readily extracted from [8, Lemma 5.11]

3.7 Lemma. *The natural map $|\mathcal{E}_\bullet|_{\mathsf{L}} \to \operatorname{hlim}_t |P_t\mathcal{E}_\bullet|_{\mathsf{L}}$ is a weak equivalence. In particular there is a short exact sequence*

$$0 \to \lim_t{}^1 \mathbb{H}^{*-1}(X; |P_t\mathcal{E}_\bullet|_{\mathsf{L}}) \to \mathbb{H}^*(X; |\mathcal{E}_\bullet|_{\mathsf{L}}) \to \lim_t \mathbb{H}^*(X; |P_t\mathcal{E}_\bullet|_{\mathsf{L}}) \to 0.$$

Associated to the tower (3.5.2) is a spectral sequence

$$E_2^{s,t}(\mathcal{E}_\bullet) = \mathbb{H}^{s-t}(X; |K(\pi_t\mathcal{E}_\bullet, t)|_{\mathsf{L}}) \tag{3.7.1}$$

converging conditionally [7, 5.10] to $\mathbb{H}^{s-t}(X; |\mathcal{E}_\bullet|_{\mathsf{L}})$ with differentials

$$d_r : E_r^{s,t}(\mathcal{E}_\bullet) \to E_r^{s+r,t+r-1}(\mathcal{E}_\bullet).$$

For $r > t+1$ we have $E_{r+1}^{s,t} \subseteq E_r^{s,t}$. Let $E_\infty^{s,t} = \cap_{r>t+1} E_r^{s,t}$. Filter $\mathbb{H}^*(X; |\mathcal{E}_\bullet|_{\mathsf{L}})$ by subgroups

$$F^t\mathbb{H}^*(X; |\mathcal{E}_\bullet|_{\mathsf{L}}) := \ker\big(\mathbb{H}^*(X; |\mathcal{E}_\bullet|_{\mathsf{L}}) \to \mathbb{H}^*(X; |P_{t-1}\mathcal{E}_\bullet|_{\mathsf{L}})\big).$$

This filtration is complete Hausdorff, and the natural map

$$\theta : \frac{F^t\mathbb{H}^*(X; |\mathcal{E}_\bullet|_{\mathsf{L}})}{F^{t+1}\mathbb{H}^*(X; |\mathcal{E}_\bullet|_{\mathsf{L}})} \to E_\infty^{*+t,t}$$

is injective. The spectral sequence is said to converge *strongly* if the map θ is an isomorphism. Equivalently the spectral sequence converges strongly iff

$$\lim_{r>t+1}{}^1 E_r^{*+t,t} = 0$$

for each t (cf. op.cit. Theorem 7.4 or [8, Prop. 6.3]).

Let $A_\bullet$ be presheaf of simplicial abelian groups on X. Associated to $A_\bullet$ is a presheaf $\mathbb{B}(A)$ of spectra (see [17, Appendix B]), and a natural isomorphism $\pi_*A_\bullet \xrightarrow{\sim} \pi_*\mathbb{B}(A)$. Let $\tilde{A}_\bullet$ denote the sheaf in the category $\mathsf{Ch}(X)$ of cochain complexes on X associated to $A_\bullet$; one may take $\tilde{A}_\bullet$ to

be the complex with the alternating sum of the face operations or one of the associated normalized complexes. There is a natural isomorphism of hypercohomology $\mathbb{H}^*(X;\tilde{A}_\bullet) \cong \mathbb{H}^*(X;\mathbb{B}(A))$ see [47, 5.32] and [28, Ch.4]. Note that we must work with chain complexes and not simplicial abelian groups, i.e., nonpositive complexes [17, Lemma 12.8].

3.8 Theorem. *Let X be an admissible space. Then there is a natural isomorphism*

$$\mathbb{H}^{s-t}(X;|K(\pi_t\mathcal{E}_\bullet,t)|_L) \simeq \mathbb{H}^s(X;\tilde{\pi}_t\mathcal{E}_\bullet)$$

and the spectral sequence (3.7.1) *converges strongly.*

Proof. Since X is an admissible space it has finite cohomological dimension, and it readily follows from Brown [12, Theorem 8] that for each t there is a strongly convergent spectral sequence

$$E_2^{p,q}(t) = H^p\big(X;\pi_q\mathbb{B}(\pi_t\mathcal{E}_\bullet)\big) \Longrightarrow \mathbb{H}^{p-q}\big(X;\mathbb{B}(\pi_t\mathcal{E}_\bullet)\big) \simeq \mathbb{H}^{p-q}(X;\tilde{\pi}_t\mathcal{E}_\bullet).$$

As well as a strongly convergent spectral sequence

$$E_2^{p,q}(t)' = H^p(X;\pi_q|K(\pi_t\mathcal{E}_\bullet,t)|_L) \Longrightarrow \mathbb{H}^{p-q}(X;|K(\pi_t\mathcal{E}_\bullet,t)|_L).$$

Since $\pi_q|K(\pi_t\mathcal{E}_\bullet,t)|_L(U) \cong \pi_{q-t}\pi_t\mathcal{E}_\bullet(U)$ (cf. [10, B5]), we obtain an isomorphism $E_2^{p,q-t}(t) \cong E_2^{p,q}(t)'$, and the proof readily follows. $\square$

4. Complex Varieties

4.1. Continuous families of analytic functions

Let X be a non-singular complex projective variety. We denote by X^{an} the topological space of closed point of X with the classical topology. Recall that X^{an} is a smooth complex analytic space. Let $\mathcal{O}^{an}$ denote the sheaf of analytic functions on X^{an}. The inclusion $\alpha : X^{an} \to X$ is continuous and induces a map $\alpha : (X^{an},\mathcal{O}^{an}) \to (X,\mathcal{O})$ of locally ringed spaces. Let $\mathcal{K}_p$ (resp. $\mathcal{K}_p^{an}$) denote the sheaf associated to the presheaf $U \mapsto K_p(\mathcal{O}(U))$ on X (resp. $U \mapsto K_p(\mathcal{O}^{an}(U))$ on X^{an}), and let $\mathcal{K}_p^{an}(\Delta^\bullet)$ denote the sheaf on X^{an} of simplicial abelian groups associated to the presheaf

$$U \mapsto K_p(\mathcal{O}^{an}_{\Delta^\bullet}(U)),$$

where $\mathcal{O}^{an}_{\Delta^p}(U) \subset \mathcal{C}_{\Delta^p}(U)$ denotes the subsheaf of rings of continuous functions $f : \Delta^p \times U \to \mathbf{C}$ such that for each $t \in \Delta^p$ the function $f_t : U \to \mathbf{C}$, where $f_t(x) := f(t,x)$, is analytic.

For F be a sheaf of simplicial abelian groups on X^{an}, let $\tilde{\pi}_n F$ denote the sheaf associated to the presheaf $U \mapsto \pi_n F(U)$. We say that a map $f : F \to G$ of sheaves of simplicial abelian groups on X^{an} is a weak equivalence if $f_* : \tilde{\pi}_n F \to \tilde{\pi}_n G$ is an isomorphism of abelian groups for $n \geq 0$.

4.2 Lemma. *Let $\mathbf{C}(\Delta^\bullet)$ denote the simplicial sheaf associated the the "constant" simplicial presheaf $U \mapsto \mathbf{C}(\Delta^\bullet)$. The map $\mathbf{C}(\Delta^\bullet) \to \mathcal{O}^{an}_{\Delta^\bullet}$ identifying $\mathbf{C}(\Delta^\bullet)$ with the constant functions is a weak equivalence of simplicial sheaves.*

Proof. Since X^{an} is a complex manifold, of complex dimension n say, each point $x \in X^{an}$ has an open neighborhood N biholomorphic to the open disc $D^n = \{z \in \mathbf{C}^n : \|z\| < 1\}$. In particular each point has a contractible neighborhood. For such a neighborhood it is not hard to construct a simplicial homotopy equivalence from the map $\varepsilon_x : \mathcal{O}^{an}_{\Delta^\bullet}|_N \to \mathcal{O}^{an}_{\Delta^\bullet}|_N$, where $\varepsilon_x f(y,t) = f(x,t)$ to the identity map. Thus $\tilde{\pi}_n \mathbf{C}(\Delta^\bullet) \to \tilde{\pi}_n \mathcal{O}^{an}_{\Delta^\bullet}|_N$ is an isomorphism, and the proof of the lemma follows. □

The map of simplicial presheaves $\mathbf{C}(\Delta^\bullet) \to \mathcal{O}^{an}_{\Delta^\bullet}$ induces a map of simplicial sheaves $i_* : K_p(\Delta^\bullet) \to \mathcal{K}^{an}_p(\Delta^\bullet)$, where $K_p(\Delta^\bullet)$ is the 'constant' simplicial sheaf associated to $K_p(\mathbf{C}(\Delta^\bullet))$. From the proof of the lemma we obtain

4.3 Theorem. *The map i_* is an weak equivalence for $p \geq 0$. In particular there is a natural isomorphism*

$$\mathbb{H}^*(X^{an}, K_p(\Delta^\bullet)) \to \mathbb{H}^*(X^{an}, \mathcal{K}^{an}_p(\Delta^\bullet)),$$

and a strongly convergent spectral sequence

$$E_2^{s,t} = \mathbb{H}^s(X^{an}; \mathcal{K}^{an}_t(\Delta^\bullet)) \Longrightarrow \boldsymbol{ku}^{s-t}(X^{an}).$$

Let W be a compact space. Since $\mathbf{C}(W)$ is a Banach algebra, the group $K_1(\mathbf{C}(W))$ splits as a direct sum of the group $\mathbf{C}(W)^*$ of units of $\mathbf{C}(W)$ and the group $[W, \mathrm{SL}^{\mathrm{top}}\mathbf{C}]$ of homotopy classes of maps from W to the special linear group $\mathrm{SL}^{\mathrm{top}}\mathbf{C}$ (see [35, §7]). In particular $K_1(\Delta^\bullet)$ is the sheaf associated to the constant simplicial presheaf $\mathbf{C}^*(\Delta^\bullet)$. Thus $K_1(\Delta^\bullet)$ is the sheaf associated to the Eilenberg-Mac Lane simplicial presheaf of type $(\mathbf{Z}, 1)$, and it follows [28, Ch.4] that

$$\mathbb{H}^*(X^{an}, \mathcal{K}^{an}_1(\Delta^\bullet)) \cong H^{*+1}(X^{an}, \mathbf{Z}(1)).$$

Let $\mathcal{K}^{an}_1(\Delta^\bullet) \to I$ be a resolution by flasque sheaves with

$$\mathcal{K}^{an}_1(\Delta^q) \to I^{0,-q} \to I^{1,-q} \to \dots$$

define the total complex

$$\operatorname{Tot}^{\prod}(I)^n = \prod_{p+q=n} I^{p,q}.$$

Let F_q denote the filtration with

$$F_q \operatorname{Tot}^{\prod}(I)^n = \prod_{\substack{p+q'=n \\ q' \geq q}} I^{p,q}. \tag{4.3.1}$$

Since X^{an} is an admissible space $H^p(X^{an}, K_1(\Delta^{-q})) = 0$ for p large. It follows that the filtration determines a spectral sequence $E_1^{p,q} = H^p(X^{an}, K_1(\Delta^{-q}))$ converging strongly to $\mathbb{H}^{p+q}(X^{an}, K_1(\Delta^\bullet))$. In addition the simplicial sheaf $\mathcal{K}_1(\Delta^\bullet)$ is weakly equivalent to $\mathcal{O}_{\Delta^\bullet}^{an*}$ and there is a commutative diagram

$$\begin{tikzcd}
H^p(X^{an}, \mathcal{K}_1(\Delta^0)) \arrow[r, "s"] \arrow[d, equal] & E_\infty^{p,0} \arrow[d, "i"] \\
H^p(X^{an}, \mathcal{O}^{an*}) \arrow[r, "\delta"] & H^{p+1}(X^{an}, \mathbf{Z}(1)),
\end{tikzcd}$$

where s surjective and i is injective.

4.4 Proposition. *The edge map*

$$H^p(X^{an}, \mathcal{K}_1^{an}(\Delta^0)) \overset{s}{\twoheadrightarrow} E_\infty^{p,0} \overset{i}{\rightarrowtail} H^{p+1}(X^{an}, \mathbf{Z})$$

is the same map that arises from the exponential sequence.

Proof. In the category $\mathsf{Spt}(X^{an})$ of presheaves of spectra on X^{an} there is a fibration sequence

$$\mathbf{Z}(1) \to \mathcal{O}_{\Delta^\bullet}^{an} \to \mathcal{O}_{\Delta^\bullet}^{an*}. \tag{4.4.1}$$

And we obtain a distinguished triangle

$$\mathbf{Z}(1) \to \mathcal{O}_{\Delta^\bullet}^{an} \to \mathcal{O}_{\Delta^\bullet}^{an*} \to \mathbf{Z}(1)[1]$$

in $\mathsf{Ho}\,\mathsf{Spt}(X^{an})$. The simplicial sheaf $\mathcal{O}_{\Delta^\bullet}^{an}$ is weakly equivalent to $\mathcal{C}_{\Delta^\bullet}$. Thus natural map $0 \to \mathcal{O}_{\Delta^\bullet}^{an}$ is a weak equivalence, and $\mathcal{O}_{\Delta^\bullet}^{an*} \to \mathbf{Z}(1)[1]$ is a weak equivalence. By applying the functor $\mathbb{B}$ to $\mathcal{K}_1^{an}$, we may view it as a sheaf of spectra on X^{an}. There is a natural map $r : \mathcal{K}_1^{an} = \mathcal{K}_1^{an}(\Delta^0) \to \mathcal{K}_1^{an}(\Delta^\bullet)$, and by pulling back the fibration sequence (4.4.1) along r, we get the distinguished exponential sequence

$$\mathbf{Z}(1) \to \mathcal{O}^{an} \to \mathcal{K}_1^{an} \to \mathbf{Z}(1)[1]$$

and a commutative diagram

$$\begin{tikzcd}
H^p(X^{an}, \mathcal{K}_1^{an}) \arrow[r] \arrow[d] & \mathbb{H}^p(X^{an}, \mathcal{K}_1^{an}(\Delta^\bullet)) \arrow[d] \\
H^{p+1}(X^{an}, \mathbf{Z}(1)) \arrow[r] & H^{p+1}(X^{an}, \mathbf{Z}(1)),
\end{tikzcd}$$

where the top horizontal map is the same map induced by the filtration (4.3.1). □

The simplicial abelian groups $K_p(\mathbf{C}(\Delta^\bullet))$ for $p > 1$ are not known explicitly; however, we do have a qualitative description of them by combining the work of Suslin [43], Gillet-Thomason [22], Gabber [18], Prasolov [38], and Fischer [16].

4.5 Theorem. *The simplicial abelian group $K_p(\mathbf{C}(\Delta^\bullet))$ is trivial for $p < 0$, and for $p \geq 0$ we have:*

- *$K_0(\mathbf{C}(\Delta^\bullet))$ is the constant simplicial group $\mathbf{Z}$.*
- *$K_{2p}(\mathbf{C}(\Delta^\bullet))$ is a simplicial rational vector space for $p > 0$.*
- *$K_{2p-1}(\mathbf{C}(\Delta^\bullet))$ is a simplicial divisible abelian group, its torsion subgroup $K_{2p-1}(\mathbf{C})_{\mathrm{tors}}$ is isomorphic to $\mathbf{Q}/\mathbf{Z}$ and*

$$K_{2p-1}(\mathbf{C}(\Delta^\bullet))/K_{2p-1}(\mathbf{C})_{\mathrm{tors}}$$

  is a simplicial rational vector space.

4.6. The Bott element

By Theorem 1.12 there is a spectral sequence $E^1_{p,q} = K_q(\mathbf{C}(\Delta^p)) \Longrightarrow \pi_{p+q}\boldsymbol{ku}$ with a pairing $E^r_{p,q} \otimes E^r_{s,t} \to E^r_{p+s,q+t}$. Let

$$\beta(t) = \exp(2\pi\sqrt{-1}t) \in K_1(\mathbf{C}(\Delta^1)).$$

The element β determines a class $[\beta] \in \pi_1 K_1(\mathbf{C}(\Delta^\bullet))$ and an isomorphism

$$\mathbf{Z}(1) \xrightarrow{\sim} E^2_{1,1} = E^\infty_{1,1} = \pi_2\boldsymbol{ku}.$$

From the multiplicative structure we get isomorphisms $\mathbf{Z}(p) \xrightarrow{\sim} E^\infty_{p,p}$. However, the spectral sequence does not degenerate. To see this we consider the E^2-terms. From the previous theorem $K_{2q}(\mathbf{C}(\Delta^\bullet)) = E^1_{\bullet,2q}$ is a chain complex of rational vector spaces for $q > 0$. Hence

$E^2_{p,2q} = \pi_p K_{2q}(\mathbf{C}(\Delta^\bullet))$ is a rational vector space. In particular $E^2_{2p,2p} \neq E^\infty_{2p,2p} \cong \mathbf{Z}(p)$ for $p > 0$. In odd dimensions we have a fibration

$$K_{2q+1}(\mathbf{C})_{\mathrm{tors}} \to K_{2q+1}(\mathbf{C}(\Delta^\bullet)) \to V,$$

where V is a simplicial rational vector space. Since $K_{2q+1}(\mathbf{C})_{\mathrm{tors}}$ is a constant simplicial abelian group, the long exact sequence of homotopy groups simplifies to the families $\pi_p K_{2q+1}(\mathbf{C}(\Delta^\bullet)) \simeq \pi_p V$ of isomorphisms and an exact sequence

$$\begin{tikzcd}
\pi_1 K_{2q+1}(\mathbf{C}(\Delta^\bullet)) \arrow[r, tail] & \pi_1 V \arrow[d] & \\
& K_{2q+1}(\mathbf{C})_{\mathrm{tors}} \arrow[r] & \pi_0 K_{2q+1}(\mathbf{C}(\Delta^\bullet)) \arrow[r, two heads] & \pi_0 V
\end{tikzcd}$$

4.7 Proposition. *The homotopy groups of* $\mathbf{C}^*(\Delta^\bullet) \otimes \mathbf{C}^*(\Delta^\bullet)$ *are*

$$\pi_p(\mathbf{C}^*(\Delta^\bullet) \otimes \mathbf{C}^*(\Delta^\bullet)) = \begin{cases} \mathbf{Q} & p = 2 \\ 0 & p \neq 2. \end{cases}$$

Proof. Consider $\mathbf{C}^*(\Delta^\bullet) \tilde{\otimes} \mathbf{C}^*(\Delta^\bullet)$ as a double chain complex with $\mathbf{C}^*(\Delta^p) \otimes \mathbf{C}^*(\Delta^q)$ in bidegree (p, q). By the familiar Dold-Kan and Eilenberg-Zilber-Cartier correspondences [33] the homology of this double complex is canonically isomorphic to the homotopy groups of the simplicial abelian group $\mathbf{C}^*(\Delta^\bullet) \otimes \mathbf{C}^*(\Delta^\bullet)$.

Let $p \geq 0$ be an integer and consider the chain complex $\mathbf{C}^*(\Delta^p) \otimes \mathbf{C}^*(\Delta^\bullet)$. First we shall show that

$$H_q(\mathbf{C}^*(\Delta^p) \otimes \mathbf{C}^*(\Delta^\bullet)) \cong \begin{cases} \mathbf{C}^*(\Delta^p) \otimes \mathbf{Q} & q = 1 \\ 0 & q \neq 1. \end{cases} \tag{4.7.1}$$

By identifying $\mathbf{Q}/\mathbf{Z}$ with $\mathbf{C}^*(\Delta^\bullet)_{\mathrm{tors}}$ we get a short exact sequence

$$\mathbf{Q}/\mathbf{Z} \rightarrowtail \mathbf{C}^*(\Delta^\bullet) \twoheadrightarrow \mathbf{C}^*(\Delta^\bullet)/\mathrm{tors}$$

of chain complexes with $\mathbf{C}^*(\Delta^\bullet)/\mathrm{tors}$ a complex of rational vector spaces. From the long exact homology sequence we see that the group $H_p(\mathbf{C}^*(\Delta^\bullet)/\mathrm{tors})$ is zero for $p \neq 1$, and for $p = 1$ we have a short exact sequence

$$0 \to \mathbf{Z} \to H_1(\mathbf{C}^*(\Delta^\bullet)/\mathrm{tors}) \to \mathbf{Q}/\mathbf{Z} \to 0.$$

Since $H_1(\mathbf{C}^*(\Delta^\bullet)/\mathrm{tors})$ is a rational vector space, $\mathbf{C}^*(\Delta^\bullet)/\mathrm{tors}$ is an Eilenberg-Mac Lane space of type $K(\mathbf{Q}, 1)$.

For each $q \geq 0$ the short exact sequence

$$\mathbf{C}^*(\Delta^q)_{\mathrm{tors}} \rightarrowtail \mathbf{C}^*(\Delta^q) \twoheadrightarrow \mathbf{C}^*(\Delta^q)/\mathrm{tors}$$

splits. Hence for any abelian group A we obtain a short exact sequence

$$A \otimes \mathbf{C}^*(\Delta^\bullet) \rightarrowtail A \otimes \mathbf{C}^*(\Delta^\bullet) \twoheadrightarrow A \otimes \mathbf{C}^*(\Delta^\bullet)/\mathrm{tors}.$$

Taking $A = \mathbf{C}^*(\Delta^p)_{\mathrm{tors}}$ and noting that $\mathbf{C}^*(\Delta^p)_{\mathrm{tors}} \otimes \mathbf{C}^*(\Delta^\bullet)/\mathrm{tors}$ and $\mathbf{C}^*(\Delta^p)_{\mathrm{tors}} \otimes \mathbf{C}^*(\Delta^\bullet)_{\mathrm{tors}}$ are trivial we see that

$$H_q\big(\mathbf{C}^*(\Delta^p)_{\mathrm{tors}} \otimes \mathbf{C}^*(\Delta^\bullet)\big) = 0,$$

and

$$\begin{aligned} H_q\big(\mathbf{C}^*(\Delta^p) \otimes \mathbf{C}^*(\Delta^\bullet)\big) &\cong H_q\big((\mathbf{C}^*(\Delta^p)_{\mathrm{tors}} \oplus \mathbf{C}^*(\Delta^p)/\mathrm{tors}) \otimes \mathbf{C}^*(\Delta^\bullet)\big) \\ &\cong \begin{cases} \mathbf{C}^*(\Delta^p) \otimes \mathbf{Q} & q = 1 \\ 0 & q \neq 1. \end{cases} \end{aligned}$$

proving the claim (4.7.1). The proof of the Proposition follows by considering the spectral sequence associated horizontal filtration of the double complex and noting that (4.7.1) gives

$$E^2_{p,q} \cong \begin{cases} \mathbf{Q} & (p,q) = (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

□

4.8 Remark. A straightforward induction shows that for $q > 1$ the simplicial abelian group $\mathbf{C}^*(\Delta^\bullet)^{\otimes q}$ is an Eilenberg-Mac Lane space of type $K(\mathbf{Q}, q)$. In particular there is a non-trivial map $K(\mathbf{Q}, q) \to K_q\big(\mathbf{C}(\Delta^\bullet)\big)$ inducing a map $1 \mapsto [\beta^q] \in \pi_q K_q\big(\mathbf{C}(\Delta^\bullet)\big)$ on homotopy.

4.9 Proposition. *There is a spectral sequence*

$$\mathbb{H}^p\big(X^{an}; \tilde{\pi}_q K_t(\Delta^\bullet)\big) \Longrightarrow \mathbb{H}^{p-q}\big(X^{an}; \mathcal{K}^{an}_t(\Delta^\bullet)\big)$$

that degenerates at the E_2 level.

Proof. The hypercohomology of X^{an} with coefficients in the constant sheaf of simplicial abelian groups $K_t(\Delta^\bullet)$, viewed as an unbounded cochain complex, is naturally isomorphic to the hypercohomology of X^{an} with coefficients in the presheaf of spectra $\mathbb{B}\big(K_t(\Delta^\bullet)\big)$. The latter theory is furnished with a descent spectral sequence. Since there is a natural isomorphism $\tilde{\pi}_t \mathbb{B}\big(K_t(\Delta^\bullet)\big) \cong \tilde{\pi}_t K_t(\Delta^\bullet)$, we obtain a strongly convergent spectral sequence as claimed.

To see that the spectral sequence degenerates, we consider first the case $t = 0$. Since $K_0(\Delta^\bullet)$ is the constant simplicial sheaf $\mathbf{Z}$ we have

$$E_2^{p,q} = H^p(X^{an}, \tilde{\pi}_q \mathbf{Z}) = \begin{cases} H^p(X^{an}, \mathbf{Z}) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose that $t > 0$. The sheaf $\tilde{\pi}_q K_{2t}(\Delta^\bullet)$ is a constant sheaf of rational vector spaces and $\tilde{\pi}_q K_{2t-1}(\Delta^\bullet)$ is a constant sheaf of the form

$$\tilde{\pi}_q K_{2t-1}(\Delta^\bullet) = \begin{cases} K_{2p+1}(\mathbf{C})_{\mathrm{tors}} \oplus V_0 & \text{if } q = 0 \\ V_q & \text{if } q > 0, \end{cases}$$

where V_q is a rational vector space. It readily follows that d_r is the trivial map for $r \geq 2$. $\square$

From the degeneration of the spectral sequence and the associated filtration we obtain

$$\frac{F^q \mathbb{H}^*(X^{an}; K_t(\Delta^\bullet))}{F^{q+1}\mathbb{H}^*(X^{an}; K_t(\Delta^\bullet))} \simeq H^{*+q}(X^{an}; \tilde{\pi}_q K_t(\Delta^\bullet)).$$

By choosing a splitting we obtain

$$\mathbb{H}^s(X^{an}; K_t(\Delta^\bullet)) \cong \bigoplus_{q \geq 0} H^{s+q}(X^{an}; \tilde{\pi}_q K_t(\Delta^\bullet)).$$

4.10. Direct images

As we described in §3.4 the pair of adjoint functors $(\alpha^\#, \alpha_\#)$ prolongs pointwise to a pair of adjoint functors

$$\mathsf{s.Spt}(X) \underset{\mathsf{s}.\alpha_\#}{\overset{\mathsf{s}.\alpha^\#}{\rightleftarrows}} \mathsf{s.Spt}(X^{an}),$$

the total derived functors $\big(\mathsf{L}(\mathsf{s}.\alpha^\#), \mathsf{R}(\mathsf{s}.\alpha_\#)\big)$ exist and form an adjoint pair. Let $\mathsf{D}(X)$ and $\mathsf{D}(X^{an})$ denote the derived categories of sheaves of abelian groups on X and X^{an} respectively, and let $\mathsf{Ch}(X)$ and $\mathsf{Ch}(X^{an})$ denote the respective categories of chain complexes. It follows from Hovey [26, §5] that $\mathsf{Ch}(X)$ and $\mathsf{Ch}(X^{an})$ are proper closed model categories in which weak equivalences are quasi-isomorphisms and fibrations are surjective maps with flasque kernels (cf. [12, §9]); in particular, $\mathsf{Ho}\,\mathsf{Ch}(X) = \mathsf{D}(X)$ and $\mathsf{Ho}\,\mathsf{Ch}(X^{an}) = \mathsf{D}(X^{an})$.[1] Furthermore, the total derived functors $\mathsf{R}\alpha_*, \mathsf{L}\alpha^*$ exist and form an adjoint

[1] Morel has most likely been aware of this see [13, Appendix C].

pair. The functor $N\tilde{\pi}_t : \mathsf{s.Spt}(X) \to \mathsf{Ch}(X)$, where N is the normalization functor, carries weak equivalences to quasi-isomorphisms and has a total derived functor $N\tilde{\pi}_t : \mathsf{Ho\,s.Ab}(X) \to \mathsf{D}(X)$.

4.11 Lemma. *The following diagram is commutative*

$$\begin{tikzcd} \mathsf{Ho\,s.Spt}(X^{an}) \arrow[r, "\mathsf{Rs.}\alpha_{\#}"] \arrow[d, "N\tilde{\pi}_t"] & \mathsf{Ho\,s.Spt}(X) \arrow[d, "N\tilde{\pi}_t"] \\ \mathsf{D}(X^{an}) \arrow[r, "\mathsf{R}\alpha_*"] & \mathsf{D}(X) \end{tikzcd}$$

up to a natural isomorphism of functors.

Proof. We begin by considering the commutative diagram

$$\begin{tikzcd} \mathsf{s.Spt}(X^{an}) \arrow[r, "\mathsf{s.}\alpha_{\#}"] \arrow[d, "N\tilde{\pi}_t"] & \mathsf{s.Spt}(X) \arrow[d, "N\tilde{\pi}_t"] \\ \mathsf{Ch}(X^{an}) \arrow[r, "\alpha_*"] & \mathsf{Ch}(X^{an}). \end{tikzcd}$$

Note, however, that taking the total right derived functor is only a lax-functor, i.e., if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are functor between closed model categories and the total derived functors $\mathsf{R}F, \mathsf{R}G$ and $\mathsf{R}(GF)$ exist, then there is a natural transformation $\tau : \mathsf{R}(GF) \stackrel{\cdot}{\to} \mathsf{R}(G)\mathsf{R}(F)$. Fortunately in the case at hand we have a natural isomorphism

$$\mathsf{R}(N\tilde{\pi}_t)\mathsf{R}(\mathsf{s.}\alpha_{\#}) \xleftarrow{\sim} \mathsf{R}(N\tilde{\pi}_t\mathsf{s.}\alpha_{\#}) = \mathsf{R}(\mathsf{s.}\alpha_* N\tilde{\pi}_t).$$

This follows from the observations that $\alpha_{\#}$ and α_* preserve fibrations and weak equivalences between fibrant objects and $\tilde{\pi}_t$ and N preserve weak equivalences see [12].

The natural transformation

$$\mathsf{R}(\mathsf{s.}\alpha_* N\tilde{\pi}_t) \to \mathsf{R}(\mathsf{s.}\alpha_*)\mathsf{R}(N\tilde{\pi}_t)$$

is also a natural isomorphism since the functor $N\tilde{\pi}_t$ carries weak equivalences to isomorphisms and α_* preserves isomorphisms. $\square$

Let $K^{an}(\Delta^\bullet)$ denote the presheaf of simplicial spectra on X^{an} given by $K^{an}(\Delta^\bullet)(U) = K(\Delta^\bullet \times U)$, and let

$$K^{Zar}(\Delta^\bullet) = \mathsf{R}(\mathsf{s.}\alpha_{\#})K^{an}(\Delta^\bullet) \in \mathsf{Ho\,s.Spt}(X).$$

Let $\mathcal{K}_p^{Zar}(\Delta^\bullet) = \mathsf{R}(\alpha_*)\mathcal{K}_p^{an}(\Delta^\bullet) \simeq \mathsf{R}(N\tilde{\pi}_p)K^{Zar}$ denote the total direct image in the homotopy category $\mathsf{D}(X)$. Recall that $K_t(\Delta^\bullet)$ denotes the

sheaf on X^{an} associated to the constant presheaf $K_t(\mathbf{C}(\Delta^\bullet))$. There are canonical maps

$$\mathcal{K}_p \to \mathcal{K}_p^{Zar}(\Delta^0) \to \mathcal{K}_p^{Zar}(\Delta^\bullet), \tag{4.11.1}$$

inducing a map

$$\phi : H^s(X;\mathcal{K}_t) \to \mathbb{H}^s(X;\mathcal{K}_t^{Zar}(\Delta^\bullet)) \cong \mathbb{H}^s(X^{an};\mathcal{K}_t^{an}(\Delta^\bullet)),$$

which specializes to the cycle class map

$$CH^1(X) \to H^2(X^{an};\mathbf{Z}(1)).$$

for $s = t = 1$ by 4.3 and 4.4.

Consider the spectral sequence

$$E_2^{s,t} = \mathbb{H}^s(X;\mathcal{K}_t^{Zar}(\Delta^\bullet)) \Longrightarrow \boldsymbol{ku}^{s-t}(X^{an}).$$

and the Brown-Gersten-Quillen spectral sequence

$${}^{BGQ}E_2^{s,t} = H^s(X,\mathcal{K}_t) \Longrightarrow K_{t-s}(X).$$

There is a natural map $\varphi : {}^{BGQ}E_2^{s,t} \to E_2^{s,t}$ of abelian groups. The maps in (4.11.1) induces a map of hypercohomology spectral sequences, and since BGQ-spectral sequence can be derived from Postnikov towers [21], we obtain:

4.12 Theorem. *The map φ prolongs to a map of multiplicative spectral sequences.*

$$\begin{array}{ccc} H^p(X,\mathcal{K}_q) & \longrightarrow & \mathbb{H}^p(X^{an};\mathcal{K}_q^{an}(\Delta^\bullet)) \\ \Downarrow & & \Downarrow \\ K_{q-p}(X) & \longrightarrow & \boldsymbol{ku}^{p-q}(X^{an}) \end{array}$$

Appendix

A. Spectra and simplicial spectra

A.1. Stable model categories

Let $\mathcal{C}$ be a pointed simplicial model category. Choose a zero, i.e., an initial and final object $o \in \mathcal{C}$. For each pair (A, B) of objects in $\mathcal{C}$ the mapping space $\mathsf{map}\,(A, B)$ is a pointed simplicial set with base point the zero map $A \to o \to B$, and if $f : A' \to A, g : B \to B'$ are maps in $\mathcal{C}$ the induced maps

$$\mathsf{map}\,(A', B) \leftarrow \mathsf{map}\,(A, B) \to \mathsf{map}\,(A, B')$$

preserve the base point in $\mathit{map}\,(A, B)$. Thus the simplicial enrichment on $\mathcal{C}$ factors through the pointed category $\mathit{s.sets}_o$.

The *fiber* and *cofiber* of a map $f : A \to B \in \mathcal{C}$ are defined to be the pull-back of $o \to B \xleftarrow{f} A$ and the push-out of $o \leftarrow A \xrightarrow{f} B$ respectively. For any finite $K \in \mathit{s.sets}$ the tensor product $K \otimes o$ is isomorphic to o. If K is pointed with base point $b \to K$ and $X \in \mathcal{C}$ we let $K \wedge X$ denote cofiber of $b \otimes X \to K \otimes X$. Let $f : A \to B \in \mathcal{C}$ be a weak equivalence between cofibrant objects, and let K be a pointed finite simplicial sets. It follows from [10, Lemma 4.5] that the induced maps $K \otimes A \to K \otimes B$ and $K \wedge A \to K \wedge B$ are weak equivalences. Thus for $K = S^1 = \Delta^1/\dot{\Delta}^1$, the simplicial circle, there is a total left derived functor, called *suspension*,

$$\Sigma := \mathsf{L}(S^1 \wedge -) : \mathit{Ho}\,\mathcal{C} \to \mathit{Ho}\,\mathcal{C}.$$

The dual of Σ is the *loops* functor Ω constructed as follows. If K is a finite pointed simplicial set with base point b and $X \in \mathcal{C}$, we denote the fiber of $X^K \to X^b$ by X_o^K, where X^K denotes the cotensor product. If K is a pointed finite simplicial set and if $f : X \to Y$ is a weak equivalence of fibrant objects in $\mathcal{C}$ the induced maps $X^K \to Y^K$ and $X_o^K \to Y_o^K$ are weak equivalences. It follows that there exists a total right derived functor $\Omega := \mathsf{R}(-)_0^{S^1} : \mathit{Ho}\,\mathcal{C} \to \mathit{Ho}\,\mathcal{C}$.

A.2 Remark. Quillen [39] shows that the functors $\Sigma, \Omega : \mathit{Ho}\,\mathcal{C} \to \mathit{Ho}\,\mathcal{C}$ exist for any pointed closed model category $\mathcal{C}$. In general ΣA and ΩX are objects in the homotopy category $\mathit{Ho}\,\mathcal{C}$. If the category $\mathcal{C}$ is also proper, and we view $\Sigma X, \Omega X$ as objects in $\mathcal{C}$, then ΣX represents the homotopy cofiber (cf. [10, §A]) of $\nabla : X \vee X \to X$ and ΩX represents the homotopy fiber (loc.cit.) of $\Delta : X \to X \times X$. If X is a cofibrant object in $\mathcal{C}$ then one may take $S^1 \wedge X$ to be ΣX as an object in $\mathcal{C}$, and when X is fibrant one may take $X_o^{S^1}$ to be ΩX.

Following Hovey [25, Ch. 7] we call a simplicial pointed closed model category $\mathcal{C}$ a *stable closed model* category if the functors $\Sigma, \Omega : \mathit{Ho}\,\mathcal{C} \to \mathit{Ho}\,\mathcal{C}$ are adjoint equivalences. It is a theorem (loc.cit.) that stable model categories are triangulated categories in the sense of Verdier [50]. The class of distinguished triangles in $\mathit{Ho}\,\mathcal{C}$ is the class of cofibre sequences. Since a sequence

$$\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

is a fibre sequence iff [25, 7.1.11]

$$\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{-\epsilon_Z h} \Sigma\Omega Z$$

is a cofibre sequence, cofibre sequences and fibre sequences determine the same class of distinguished triangles in a stable model category. Note that Hovey does not require a simplicial structure to define stable model categories.

A.3. Spectra

We take the old-fashioned definition of a spectrum E to consist of a sequence of pointed topological spaces $E_0, E_1, \ldots$ together with pointed structures maps $\sigma : S^1_{\mathsf{top}} \wedge E_n \to E_{n+1}$. A map $f : E \to F$ of spectra consists of a sequence of pointed maps $f_n : E_n \to F_n$ such that $\sigma(1 \wedge f_n) = f_{n+1}\sigma$. We denote the category of spectra by Spt. If we replace the term topological space with simplicial set and S^1_{top} with the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$ we get the category $\mathsf{s.s.Spt}$ of $\mathsf{s.s.}$-spectra. Both these categories have proper closed model structures [10], and the singular complex functor $\mathrm{Sin} : \mathsf{Spt} \to \mathsf{s.s.Spt}$ has a left adjoint, the geometric realization functor $|\ \ |$. A $\mathsf{s.s.}$-spectrum X is fibrant if X_n is a Kan complex for each n and the adjoint map $\sigma^\sharp : X_n \to \Omega X_{n+1}$ is a weak equivalence. The two functors provide an adjoint equivalence of homotopy categories $\mathsf{Ho}\,(\mathsf{Spt})$, $\mathsf{Ho}\,(\mathsf{s.s.Spt})$.

Let Stab denote the stable homotopy category of spectra described by Adams [2, III]. The categories $\mathsf{Ho}\,(\mathsf{Spt})$ and $\mathsf{Ho}\,(\mathsf{s.s.Spt})$ are both equivalent to Stab. If E is a $\mathsf{s.s.}$-spectrum then $|E|$ is a spectrum with each $|E_n|$ a CW-complex, however the cellular maps $S^1 \wedge |E_n| \to |E_{n+1}|$ are not necessarily isomorphisms onto a subcomplex of $|E_{n+1}|$. To get a functor $\mathsf{s.s.Spt} \to \mathsf{Stab}$, one may compose $|\ \ |$ with the telescope functor tel of Adams op.cit. p. 171. It is a straight forward exercise in the yoga of closed model categories to show that $E \mapsto \mathrm{tel}\,|E|$ induces an equivalence of categories $\mathsf{Ho}\,(\mathsf{s.s.Spt}) \to \mathsf{Stab}$. Similarly there is a functor $E \mapsto \mathrm{tel}\,|\,\mathrm{Sin}\,E|$. For a spectrum E we denote the n shift suspension of E as either $\Sigma^n E$ or $E[n]$.

A.4. Simplicial spectra

For X a simplicial spectrum the n^{th}-skeleton of X is the simplicial spectrum $\mathrm{sk}_n(X)$ with

$$\mathrm{sk}_n(X)_p = \operatorname*{colim}_{\substack{[p]\to[q]\\ q\le n}} X_q,$$

and the n^{th}-coskeleton of X is the simplicial spectrum $\mathrm{csk}_n(X)$ with

$$\mathrm{csk}_n(X)_p = \lim_{\substack{[q]\to[p]\\ q\le n}} X_q.$$

We call a map $f : X \to Y$ between simplicial spectra a weak equivalence is $f_p : X_p \to Y_p$ is a weak equivalence for each $p \geq 0$; a fibration if for every $p \geq 0$ the map

$$X_p \to \mathrm{csk}_{p-1}(X)_p \times_{\mathrm{csk}_{p-1}(Y)_p} Y_p$$

is a fibration; and a cofibration if for every $p \geq 0$ the map

$$\mathrm{sk}_{\,p-1}(Y)_p \coprod_{\mathrm{sk}_{\,p-1}(X)_p} X_p \to Y_p$$

is a cofibration. The following theorem is a consequence of a more general theorem due to C. Reedy [40].

A.5 Theorem (Reedy). *The category of simplicial spectra is a proper closed model category.*

Reedy does not actually consider *proper* closed model categories. To extend the Reedy's prove to include this extra condition we first note that for any simplicial spectrum X, the n-skeleton $\mathrm{sk}_{\,n}(X)_k$ is the pushout of

$$\mathrm{sk}_{\,n-1}(X)_k \leftarrow \coprod_S \mathrm{sk}_{\,n-1}(X)_n \to \coprod_S X_n,$$

and the n-coskeleton $\mathrm{csk}_n(X)_k$ is the pullback of

$$\mathrm{csk}_{n-1}(X)_k \to \prod_D \mathrm{csk}_{n-1}(X)_n \leftarrow \prod_D X_n$$

where the S is the set of surjective maps $[k] \to [n] \in \Delta$ and D is the set of injective maps $[n] \to [k] \in \Delta$. Reedy's patching lemma [10, 3.8] implies that if $f : X \to Y$ is a fibration (resp. cofibration) of simplicial objects, then $\mathrm{csk}_n(X)_k \to \mathrm{csk}_n(Y)_k$ (resp. $\mathrm{sk}_{\,n}(X)_k \to \mathrm{sk}_{\,n}(Y)_k$) are fibrations (resp. cofibrations) for all $n, k \geq 0$. Thus $X_k \to Y_k$ is a fibration (resp. cofibration), and the property of being proper follows readily.

Let X be a cofibrant simplicial spectrum. Put $|X|_0 = X_0$, and define $|X|_n$ to be the pushout of the following diagram.

$$\begin{array}{ccc}
(\Delta^n \ltimes \mathrm{sk}_{\,n-1}X_n) \coprod_{(\partial\Delta^n \ltimes \mathrm{sk}_{\,n-1}X_n)} (\partial\Delta^n \ltimes X_n) & \longrightarrow & \Delta^n \ltimes X_n \\
\downarrow & & \\
|X|_{n-1} & &
\end{array}$$

There is a family

$$|X|_{n-1} \to |X|_n \to \Sigma^n(X_n/\mathrm{sk}_{\,n-1}(X)_n) \to \Sigma^1|X|_{n-1} \qquad \text{(A.5.1)}$$

of cofibration sequences, and the geometric realization $|X|$ is $\mathrm{colim}\,|X|_n$. We have (cf.[10])

A.6 Theorem. *The realization functor $|-|$ from the category of simplicial spectra to Spt preserves cofibrations and carries weak equivalences of cofibrant simplicial spectra to weak equivalences. In particular a total left-derived functor $|-|_{L} : \mathsf{Ho}\,(\mathsf{Spt}^{\Delta^{op}}) \to \mathsf{Stab}$ exists.*

A.7. A homotopy spectral sequence

Let X be a cofibrant simplicial spectrum. Associated to the family (A.5.1) is an exact couple

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & A_{s-1}(X) \arrow[rr, "i"] & & A_s(X) \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\
& & E_s(X) \arrow[ul, "k", "{[1]}"'] & &
\end{tikzcd} \tag{A.7.1}$$

with $A_s(X) = \oplus_t A_{s,t}(X)$ and $E_s(X) = \oplus_t E_{s,t}(X)$, where $A_{s,t}(X) = \pi_t|X|_s$ and $E_{s,t}(X) = \pi_t \Sigma^s (X_s/\mathrm{sk}_{\,s-1}(X)_s)$. Associated to this exact couple is a strongly convergent spectral sequence with (cf. [28, Corollary 4.22])

$$\pi_s(\pi_{t-s}X) \cong E^2_{s,t}(X) \Longrightarrow \pi_t|X|$$

and differentials $d_r : E^r_{s,t}(X) \to E^r_{s-r,t-1}(X)$. We filter the graded abelian group $\pi_*|X| \cong \operatorname{colim}_s A_s(X)$ by the subgroups

$$F_s\pi_*|X| := \mathrm{im}\big[A_s(X) \to \pi_*|X|\big].$$

The spectral sequence converges strongly since there are natural short exact sequences

$$0 \to F_{s-1}\pi_*|X| \to F_s\pi_*|X| \to E^\infty_s(X) \to 0,$$

and the filtration is complete Hausdorff.

A.8 Remark. Note that a shift of the grading by $\tilde{E}_{s,t}(X) := E_{s,s+t}(X)$ and $\tilde{A}_{s,t}(X) := A_{s,s+t}(X)$ gives a spectral sequence with the more familiar differentials $d_r : \tilde{E}^r_{s,t} \to \tilde{E}^r_{s-r,t+r-1}$ and

$$\pi_s(\pi_t X) \cong \tilde{E}^2_{s,t} \Longrightarrow \pi_{s+t}|X|.$$

Clearly convergence is unaffected. For X not cofibrant the relation between $E^\infty_s(X)$ and $\pi_*|X|$ is not clear.

In order to discuss pairings of spectral sequences of this type, we shall mimic the the procedure of Bousfield and Kan [11] and introduce

A.9. Models

Let S denote the sphere spectrum, and choose an initial-final spectrum o. For $s \geq 0$ let M_s^1 denote the simplicial spectrum

$$(M_s^1)_p = \begin{cases} o & \text{for } 0 \leq p < s \\ S & \text{for } p = s-1 \\ S \bigvee \vee_{i=0}^{s-1} S & \text{for } p = s \\ \vee_{D(s-1,p)} S \bigvee \vee_{D(s,p)} S & \text{for } p \geq s+1 \end{cases}$$

where $D(q,p)$ is the set of surjective maps $[p] \to [q] \in \Delta$. We define face maps $d_i : (M_s^1)_s \to (M_s^1)_{s-1}$, for $i > 0$, by mapping the non-degenerate factor S to the base point, and for $i = 0$ by mapping the same factor to $(M_s^1)_{s-1}$ by the identity. Each M_s^1 is cofibrant. Let $D_s^1 = \mathrm{tel}\, \overline{\Omega} |\operatorname{Sin} M_s^1|$, where $\overline{\Omega} X_n = \operatorname{colim}_k \Omega_k X_{n+k}$. Thus D_s^1 is a fibrant simplicial spectrum. Let $q \in \mathbf{Z}$ and consider the simplicial abelian group $\pi_q D_s^1$. The normalized chain complex $N\pi_q D_s^1$ is trivial in dimensions $p < s-1$; in dimensions $s-1$ and s we have $N\pi_q(D_s^1)_{s-1} = \pi_q S = N\pi_q(D_s^1)_s$; and $d = d_0 : N\pi_q(D_s^1)_s \to N\pi_q(D_s^1)_{s-1}$ is the identity.

Let $i \in N\pi_0(D_s^1)_s$ denote the class of the inclusion $S \to D_s^1$ onto the non-degenerate factor, and let $j \in N\pi_0(D_s^1)_{s-1}$ denote the class of the inclusion $S \to (D_s^1)_{s-1}$ by the identity.

A.10 Lemma. *Let X be a fibrant simplicial spectrum. The map*

$$[D_s^1, X]_t \to N\pi_t(X)_s$$

given by sending $f : D_s^1[t] \to X$ to $f_*(i)$ is an isomorphism.

Proof. Let X be a simplicial spectrum. The functor $X \mapsto X_s$ from the category of simplicial spectra to the category Spt has a left adjoint $Y \mapsto \Delta[s] \tilde{\ltimes} Y$, where

$$(\Delta[s] \tilde{\ltimes} Y)_q = \vee_{\Delta[s]_q} Y.$$

The functor $X \mapsto X_s$ preserves fibrations and acyclic fibrations (cf. [10, 3.7]), hence the total derived functors $L\Delta[s]$ and $R(\ \ )_s$ exist and form an adjoint pair of functors between the respective homotopy categories. Hence, for X a fibrant simplicial spectrum we obtain an isomorphism $[\Delta[s] \tilde{\ltimes} S^t, X] \cong [S^t, X_s]$.

There is a cofibration $\alpha : \Lambda^0[s-1]\tilde{\ltimes} S \to \Delta[s]\tilde{\ltimes} S$, where $\Lambda^0[s-1]$ is the union of the coface maps $d_i : \Delta[s-1] \to \Delta[s]$ for $i > 0$, and D^1_s is the homotopy cofiber of α. Thus there is a long exact sequence

$$\cdots \to [D^1_s, X]_t \to \pi_t X_s \xrightarrow{\alpha_*} [\Lambda^0[s-1]\tilde{\ltimes} S, X]_t \to [D^1_s, X]_{t-1} \to \cdots \tag{A.10.1}$$

The face maps d_i, $i > 0$ induce an injective map $[\Lambda^0[s-1]\tilde{\ltimes} S, X]_t \to \oplus_{i=1}^s \pi_t X_{s-1}$, and the degeneracy maps s_i, $0 \le i \le s$ induce a section of α_*. Thus the sequence (A.10.1) is short exact, and it follows that $[D^1_s, X]_t \cong N\pi_t X_s$. □

For $s > 0$ let $\psi : D^1_{s-1} \to D^1_s$ be a map in $N\pi_0(D^1_s)_{s-1} \cong \mathbf{Z}$ representing the generator j, and define D^2_s such that

$$D^1_{s-1} \xrightarrow{\psi} D^1_s \xrightarrow{\alpha} D^2_s \to D^1_{s-1}[1]$$

is a cofibration sequence, where D^2_s is a fibrant-cofibrant simplicial spectrum. Inductively we define cofibration sequences

$$D^1_{s-r}[r-1] \xrightarrow{\psi} D^r_s \xrightarrow{\alpha} D^{r+1}_s \to D^1_{s-r}[r] \tag{A.10.2}$$

$$D^1_s \xrightarrow{\alpha^{r-1}} D^r_s \to D^{r-1}_{s-1}[1] \to D^1_s[1] \tag{A.10.3}$$

and for $r > s$ we set D^r_s equal to the constant simplicial spectrum o.

For a simplicial spectrum X we set $D^r_s(X)_t = [D^r_s, X]_t$, and define relations $d_r : D^1_s(X)_t \rightharpoonup D^1_{s-r}(X)_{t+r-1}$ by

$$d_r := \mathrm{im}\big[D^r_s(X)_t \xrightarrow{(\alpha^{r-1}[t],\psi)} D^1_s(X)_t \times D^1_{s-r}(X)_{t+r-1}\big].$$

A.11 Lemma. *The relations d_r have the following properties:*

(i) d_r is natural and additive

(ii) d_1 is the obvious map

(iii) the domain of d_r is the kernel of d_{r-1} for $r > 1$

(iv) the indeterminacy of d_r is the image of d_{r-1} for $r > 1$

(v) d_r is a differential.

Proof. The properties are consequences of the sequences (A.10.2) and (A.10.3) (cf. [9, §12]). □

It follows that there exists a spectral sequence $\hat{E}^1_{p,q}(X) = D^1_p(X)_q$ and differentials $d_r : \hat{E}^r_{p,q}(X) \to \hat{E}^r_{p-r,q+r-1}(X)$ with $\hat{E}^{r+1}_{p,q}(X)$ the homology of the sequence

$$\hat{E}^r_{p+r,q-r+1}(X) \to \hat{E}^r_{p,q}(X) \to \hat{E}^r_{p-r,q+r-1}(X).$$

Since the homology of a chain complex derived from a simplicial abelian group is naturally isomorphic to the homology of the normalized chain complex, it follows that there is a spectral sequence with $\hat{E}^1_{p,q} = \pi_p X_q$ and whose higher terms are those given above.

We address convergence in a similar manner by constructing simplicial spectra N_s for $s \geq 0$

$$\left(N_s\right)_p = \begin{cases} o & \text{for } 0 \leq p < s \\ S & \text{for } p = s \\ \bigvee_{D(s,p)} S & \text{for } p \geq s. \end{cases}$$

Let $N_s \to B_s$ be a weak equivalence to a fibrant-cofibrant simplicial spectra. There are cofiber sequences

$$B_{s-1} \xrightarrow{k} D^1_s \xrightarrow{j} B_s \xrightarrow{i} B_{s-1}[1] \tag{A.11.1}$$

and diagrams

$$\begin{tikzcd}
 & & & B_{s-r-1}[r] \arrow[dr] & & & \\
 & & B_{s-r}[r-1] \arrow[ur] \arrow[dr] & & D^{r+1}_s \arrow[ddrr] & & \\
 & & & D^r_s \arrow[ur, "\alpha"] \arrow[drrr] & & & \\
D^1_{s-r}[r-1] \arrow[uurr] \arrow[urrr, "\psi"] & & & & & & B_s
\end{tikzcd}$$

of cofiber sequences.

A.12 Lemma. *There is a weak equivalence $S[s] \to |B_s|$ and a natural maps*

$$\begin{tikzcd}
{[B_{s-1}, X]_{*+1}} \arrow[r, "i^*"] \arrow[d] & {[B_s, X]_*} \arrow[d] \\
\pi_{*+s}|X|_{s-1} \arrow[r] & \pi_{*+s}|X|_s.
\end{tikzcd}$$

Proof. The filtration of $|N_s|$ described in (A.5.1) has one non-trivial step

$$\begin{tikzcd} & \dot\Delta^s \ltimes S \arrow[r] \arrow[d] & \Delta^s \ltimes S \arrow[d] & \\ o \arrow[r, equal] & {|N_s|_{s-1}} \arrow[r] & {|N_s|_s} \arrow[r, equal] & {|N_s|} \end{tikzcd}$$

Thus $|N_s|$ is the cofiber of $\dot\Delta^s \ltimes S \to \Delta^s \ltimes S$ and there are weak equivalences

$$|B_s| \xleftarrow{\sim} |N_s| \xrightarrow{\sim} \frac{\Delta^s}{\partial\Delta^s} \wedge S \cong S[s].$$

The simplicial spectrum N_s is completely determined by its image $\tau_s N_s$ in the truncated category $[(\Delta^{\leq s})^{op}, \mathit{Spt}]$. In fact $N_s = \mathrm{sk}_s(N)_s$. We thus obtain a natural isomorphism

$$s.\mathit{Spt}(N_s, X) = [(\Delta^{\leq s})^{op}, \mathit{Spt}](\tau_s N_s, \tau_s X).$$

Let $|-| : [(\Delta^{\leq s})^{op}, \mathit{Spt}] \to \mathit{Spt}$ denote the functor given by

$$|Y| := \int^{p\leq s} \Delta^p \ltimes Y_p\,.$$

There are natural weak equivalences $|X|_s \cong |\tau_s X|$ and $S[s] \simeq |N_s| \simeq |B_s|$. It readily follows that the realization of a map $f : N_s \to X$ is naturally the composition of the two maps $|\tau_s f| : S[s] \to |X|_s$ and $|X|_s \to |X|$.

The category $[(\Delta^{\leq s})^{op}, \mathit{Spt}]$ also carries a closed model structure, and the truncation functor τ_s preserves weak equivalences, fibrations and cofibrations. Thus the total left and right derived functors of τ_s are the same and essentially given by $Y \mapsto \tau_s Y$. In particular there are natural maps of graded abelian groups

$$[B_s, X]_* \cong [N_s, X]_* \to [N_s, \tau_s X]_* \to [|N_s|, |\tau_s X|]_* \cong \pi_{*+s}|X|_s.$$

$\square$

We set $B_s(X)_t := [B_s, X]_t$. Associated to the cofiber sequences (A.11.1) is the following exact couple.

$$\begin{tikzcd} \cdots \arrow[r, "i^*"] & B_{s-1}(X)_{*+1} \arrow[r, "i^*"] & B_s(X)_* \arrow[r, "i\ *"] \arrow[dl, "j\ *"] & \cdots \\ & D^1_s(X)_* \arrow[u, "k^*", "{[1]}"'] & & \end{tikzcd}$$

Let $B_\infty(X)_* = \operatorname{colim}_s B_s(X)_{*-s}$. We filter $B_\infty(X)_*$ by the subgroups

$$F_s B_\infty(X)_* := \operatorname{im}\big[B_s(X)_{*-s} \to B_\infty(X)_*\big].$$

Combining Lemmas A.10, A.12, [28, Lemma 4.15] and [7, Theorem 5.3] we obtain

A.13 Theorem. *Let X be a cofibrant simplicial spectrum. There is a natural map from the previous exact couple to the exact couple* (A.7.1), *and an isomorphism $\hat{E}^r_{s,t}(X) \to \tilde{E}^r_{s,t}(X)$ of spectral sequences. Moreover, there is an isomorphism $B_\infty(X)_* \to \pi_*|X|$ of filtered groups.*

A.14. Pairings

Let E and F be CW spectra. We denote their smash product [2, III.4] by $E \wedge F$. It is a CW spectrum natural in E and F, but it only has commutative, associative properties up to coherent natural weak equivalences. Nevertheless we may define the smash product of simplicial CW spectra X and Y as the simplicial CW spectra $[p] \mapsto X_p \wedge Y_p$.

A.15 Proposition. *Let X and Y be simplicial CW spectra. There are pairings for $r \geq 1$*

$$E^r_{p,q}(X) \otimes E^r_{s,t}(Y) \to E^r_{p+s,q+t}(X \wedge Y)$$

such that

(a) $d_r(x \wedge y) = d_r(x) \wedge y + (-1)^{p+q} x \wedge d_r(y)$,

(b) the pairing on E^1 is induced by the Eilenberg-Zilber shuffle product,

(c) the pairing on E^r induces the pairing on E^{r+1}.

(d) the pairing is compatible with the filtration in the sense that there are maps $F_p\pi_q|X| \otimes F_s\pi_t|Y| \to F_{s+p}\pi_{q+t}|X \wedge Y|$.

Proof. It is enough to consider the cases $X = D^r_p[q], Y = D^r_s[t]$ for $r > 1$ and $X = B_p[q], Y = B_s[t]$. For the first case it is enough to show

(i) $E^2_{p+s,q+t}(X \wedge Y) \cong E^2_{p,q}(X) \otimes E^2_{s,t}(Y)$,

(ii) for $0 < i < r$ the group $E^2_{p+s-i,q+t+i-1}(X \wedge Y)$ is trivial,

(iii) in bi-degrees $(p+s-r, q+t+r-1)$ it is isomorphic to

$$\big(E^2_{p-r,q+r-1}(X) \otimes E^2_{s,t}(Y)\big) \oplus \big(E^2_{p,q}(X) \otimes E^2_{s-r,q+r-1}(Y)\big),$$

(iv) $d_r : E^r_{p+s,q+t}(X \wedge Y) \to E^r_{p+s-r,q+t+r-1}(X \wedge Y)$ is compatible with these isomorphisms.

For K a pointed simplicial set, with base point $o \to K$, and E a spectrum, let $K\tilde{\wedge}E$ denote the simplicial spectrum $[p] \mapsto \vee_{K_p - o} E$. Let $\overline{X}$ denote the simplicial spectrum given by the the push-out of the diagram

$$X \leftarrow S^{p-r}\tilde{\wedge}S[q+r-1] \to S^{p-r}\tilde{\wedge}CS[q+r-1],$$

where S^{p-r} is the simplicial $p-r$-sphere $\Delta[p-r]/\partial\Delta[p-r]$ and $CS[q+r-1]$ is the cone on $S[q+r-1]$. Similarly we define $\overline{Y}$ by putting a cone on $S[t+r-1]$ in simplicial dimension $s-r$. There are weak equivalences $\overline{X} \cong S^p\tilde{\wedge}S[q]$ and $\overline{Y} \cong S^s\tilde{\wedge}S[t]$, and a cofibration $X \wedge Y \to \overline{X} \wedge \overline{Y}$ of simplicial spectra. From the remarks following A.5 there is a cofibration $X_p \wedge Y_p \to \overline{X}_p \wedge \overline{Y}_p$ for each simplicial dimension p, and we get a short exact sequence

$$0 \to \pi_n(\overline{X} \wedge \overline{Y}, X \wedge Y) \to \pi_{n-1}(X \wedge Y) \to \pi_{n-1}(\overline{X} \wedge \overline{Y}) \to 0$$

of simplicial abelian groups. From the weak equivalences

$$\overline{X} \wedge \overline{Y} \cong (S^p\tilde{\wedge}S[q]) \wedge (S^s\tilde{\wedge}S[t]) \cong (S^p \wedge S^s)\tilde{\wedge}S[q+t]$$

and the fact that $\pi_m(\pi_n(K\tilde{\wedge}S[k])_\bullet) = \widetilde{H}_m(K, \pi_{n-k}S)$, we see that the simplicial abelian groups $\pi_n(\overline{X} \wedge \overline{Y}) = \pi_n((S^p \wedge S^s)\tilde{\wedge}S[q+t])$ and $\pi_n(S^{p+s}\tilde{\wedge}S[q+t])$ are weakly equivalent.

To calculate $\pi_*(\overline{X} \wedge \overline{Y}/X \wedge Y)$ we consider the cofiber sequences

$$\frac{\overline{X}_n}{X_n} \wedge Y_n \to \frac{\overline{X}_n \wedge \overline{Y}_n}{X_n \wedge Y_n} \to \overline{X}_n \wedge \frac{\overline{Y}_n}{Y_n} \quad \text{for } n \geq 0 \tag{A.15.1}$$

obtained from the sequence $X_n \wedge Y_n \to \overline{X}_n \wedge Y_n \to \overline{X}_n \wedge \overline{Y}_n$ of cofibrations (cf. [2, III Lemma 9.9]). The cofiber sequence (A.15.1) is weakly equivalent to

$$S^{p-r}_n\tilde{\wedge}Y_n[q+r] \to \frac{\overline{X}_n \wedge \overline{Y}_n}{X_n \wedge Y_n} \to (S^p_n \wedge S^{s-r}_n)\tilde{\wedge}S[q+t+r].$$

Since $Y[q+r] = D^r_s[t+q+r]$ it follows that $\pi_* \frac{\overline{X}_n \wedge \overline{Y}_n}{X_n \wedge Y_n}$ is trivial for $* < t+q+r$, and there is a short exact sequence

$$\tilde{\mathbf{Z}}(S^s_* \wedge S^{p-r}_*) \rightarrowtail \pi_{q+t+r}\left(\frac{\overline{X}_* \wedge \overline{Y}_*}{X_* \wedge Y_*}\right) \twoheadrightarrow \widetilde{\mathbf{Z}}(S^p_* \wedge S^{s-r}_*)$$

of simplicial abelian groups, where $\widetilde{\mathbf{Z}}K$ is the (reduced) free abelian groups generated by K minus its base point. Thus

$$E^2_{p+s-r.q+t+r-1}(X \wedge Y) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

The first case readily follows.

Since there are weak equivalences $B_s \simeq S^s \tilde{\wedge} S$ for $s \geq 0$ the second case is easily verified. This completes the proof of the proposition. $\square$

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