Stacks and the homotopy theory of simplicial sheaves

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Introduction

This paper grew from a personal need for a collection of technical statements about homotopy theoretic objects related to stacks in connection with a project on transfer maps for presheaves of spectra. This project is briefly described in Example 18 below: there is a very general pairing which is defined for finite maps of integral Noetherian schemes and arbitrary presheaves of spectra on the category of schemes, and this map specializes to a transfer in the case where a certain stack has a global section.

The present paper defines stacks (from several points of view) as sheaves of groupoids satisfying an effective descent condition, and then discusses the basic homotopy theoretic properties of their associated classifying simplicial sheaves. It is shown that any sheaf of groupoids G has a stack completion map $\eta : G \to \mathrm{St}(G)$ such that $\mathrm{St}(G)$ is a stack (Lemma 9), and that the induced map $\eta : BG \to BG\,\mathrm{St}(G)$ of classifying simplicial sheaves is a local weak equivalence (Lemma 7). The stack completion can be constructed either geometrically (following Giraud [4]) by putting in the missing cocycles, or by using the strong stack completion functor of Joyal and Tierney [11]. Either way, the object $B\,\mathrm{St}(G)$ satisfies descent, which means that every globally fibrant model $B\,\mathrm{St}(G) \to X$ in the simplicial presheaf category is a pointwise weak equivalence in the sense that all maps $B\,\mathrm{St}(G)(U) \to X(U)$ of sections are weak equivalences of simplicial sets (Theorem 6). The Joyal-Tierney strong stack completion $G^{\wedge}$ of a sheaf of groupoids G, in fact, produces a simplicial sheaf $BG^{\wedge}$ which is globally fibrant on the nose. More generally, there is a case to be made for defining a stack to be a presheaf of groupoids G such that BG satisfies descent in the above sense. These results are proved in Sections 2 and 3 of this paper, after an identification of the topos theoretic notion of an internal equivalence $G \to H$ of sheaves of groupoids with a morphism which induces a local weak equivalence $BG \to BH$ of simplicial sheaves (Lemma 1) in Section 1.

Section 4 contains homotopy classification results which expand on the identification of the non-abelian cohomology object $H^1(*, G)$ for sheaves of groups

G with the set morphisms $[*, BG]$ from the terminal object to BG in the homotopy category of simplicial sheaves [8]. Torsors for a sheaf of groupoids G are defined by internal functors $Y \to BG$ such that Y is weakly equivalent to a point. An internal functor is defined here to be a simplicial sheaf map such that Y is given by a homotopy colimit over G in each section. From this point of view, it is essentially no surprise that the set of path components of the category of G-torsors is isomorphic to $[*, BG]$ for arbitrary sheaves of groupoids G (Theorem 14).

In the case where G is a sheaf of groups and N is a sheaf carrying a G-action, Theorem 16 identifies path components of the quotient stack with morphisms $[*, EG \times_G N]$ in the homotopy category taking values in the associated Borel construction, or classifying space of the translation groupoid. This result is exactly the technical device that is required for the transfer project, and it specializes to results of Breen [2] which classify various classes of bitorsors. The result is proven by means of a direct cocycle argument.

This paper is certainly not the end of the story about stacks and homotopy theory. There is no attempt, for example, to discuss higher non-abelian cohomology objects, and the range of applications given here is very focussed. Stacks have been vigourously studied for many years, and the results of this paper are not particularly new in the sense that most of them are known in some other form. The synthesis presented here, however, is original and the results of this paper have not yet been collected together elsewhere.

1 Internal equivalences

Suppose, for the moment that $f : G \to H$ is an ordinary functor of small groupoids. Then it's well known and easy to see that the functor f induces a weak equivalence $f_* : BG \to BH$ of the associated classifying simplicial sets if and only if f is an equivalence of categories. Put a different way, this means that the following two conditions hold:

1) the diagram of functions

$$\begin{tikzcd} \mathrm{Arr}(G) \arrow[r, "f"] \arrow[d, "{(s,t)}"'] & \mathrm{Arr}(H) \arrow[d, "{(s,t)}"] \\ \mathrm{Ob}(G) \times \mathrm{Ob}(G) \arrow[r, "f \times f"'] & \mathrm{Ob}(H) \times \mathrm{Ob}(H) \end{tikzcd} \tag{1}$$

is a pullback, and

2) for every object x of H there is an object y of G and a morphism $x \to f(y)$ in H.

Here, $s = d_1$ is the source map and $t = d_0$ is the target.

The first condition says that the functor f is fully faithful, while the second says that objects of H should lift to G up to isomorphism. In the context of classifying spaces, the first condition says simultaneously that f induces a monomomorphism $\pi_0 BG \to \pi_0 BH$ and an isomorphism on $\pi_1(BG, x) \to \pi_1(BH, f(x))$ on all possible automorphism groups, while the second says that the map $\pi_0 BG \to \pi_0 BH$ is surjective. Every groupoid is equivalent to a disjoint union of groups, whose classifying spaces have no higher homotopy groups, so that these two conditions are enough to show that $BG \to BH$ is a weak equivalence.

Condition 2) above can be tranlated into diagram theoretic terms. Consider the picture

$$\begin{tikzcd} \mathrm{Ob}(G) \times_{\mathrm{Ob}(H)} \mathrm{Arr}(H) \arrow[r, "f_*"] \arrow[d, "s_*"'] & \mathrm{Arr}(H) \arrow[r, "t"] \arrow[d, "s"] & \mathrm{Ob}(H) \\ \mathrm{Ob}(G) \arrow[r, "f"'] & \mathrm{Ob}(H) & \end{tikzcd}$$

Then condition 2) above is equivalent to the following:

2') the composite function

$$\mathrm{Ob}(G) \times_{\mathrm{Ob}(H)} \mathrm{Arr}(H) \xrightarrow{f_*} \mathrm{Arr}(H) \xrightarrow{t} \mathrm{Ob}(H) \tag{2}$$

is surjective.

Conditions 1) and 2') are easily translated into the topos theoretic setting, as has been done in several places [3], [11]. Say that a map $f : G \to H$ of sheaves of groupoids (on a small site $\mathcal{C}$) is an *internal equivalence* if the diagram (1) is a pullback of sheaves, and the composite map (2) is an epimorphism of sheaves. Note that (2) is an epimorphism if and only if for every section $x \in \mathrm{Ob}(H)(U)$ there is a covering sieve R for U such that $\phi^*(x)$ is isomorphic to an object in the image of the functor $G(V) \to H(V)$ for every $\phi : V \to U$ in R.

Lemma 1. *Suppose that $f : G \to H$ is a morphism of sheaves of groupoids. Then the map f induces a local weak equivalence of simplicial sheaves $f_* : BG \to BH$ if and only if f is an internal equivalence.*

Proof. Condition 1) is equivalent to the assertion that f is fully faithful in the sense that the induced maps $f : G(x, y) \to H(fx, fy)$ is an isomorphism of sheaves on $\mathcal{C} \downarrow U$ for all $U \in \mathcal{C}$ and for all $x, y \in \mathrm{Ob}(G)(U)$. This means that the map $G(x, y)(U) \to H(fx, fy)(U)$ is a bijection for all $U \in \mathcal{C}$ and all $x, y \in \mathrm{Ob}(G)(U)$.

Suppose that $f_* : BG \to BH$ is a weak equivalence of simplicial sheaves. Then all induced maps $f : G(x, x) \to H(fx, fx)$ are isomorphisms of sheaves on $\mathcal{C} \downarrow U$ for all $U \in \mathcal{C}$ and for all $x \in \mathrm{Ob}(G)(U)$. If $G(x, y)(U) \neq \emptyset$, then there

is a morphism $\alpha : x \to y$ in $G(U)$ and there is a commutative diagram

$$\begin{tikzcd} G(x,x) \arrow[r, "f_*"] \arrow[d, "\alpha_*"', "\cong"] & H(fx,fx) \arrow[d, "\cong"', "f(\alpha)_*"] \\ G(x,y) \arrow[r, "f_*"'] & H(fx,fy) \end{tikzcd}$$

so that the induced map $f_* : G(x,y) \to H(fx,fy)$ is an isomorphism of sheaves. If $H(fx,fy)(U) \neq \emptyset$, then since $\pi_0 BG \to \pi_0 BH$ is an isomorphism of sheaves there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x)$ and $\phi^*(y)$ are in the same component of $G(V)$ for all $\phi : V \to U$ in R. It follows that $G(x,y)(V) \neq \emptyset$ for all $\phi \in R$, and the isomorphisms

$$G(x,y)(V) \xrightarrow{f} H(fx,fy)(V)$$

induce an isomorphism $G(x,y)(U) \to H(fx,fy)(U)$. If $H(fx,fy)(U) = \emptyset$ then $G(x,y)(U) = \emptyset$. It follows that $G(x,y)(U) \to H(fx,fy)(U)$ is an isomorphism for all $x, y \in \mathrm{Ob}(G)(U)$ and for all $U \in \mathcal{C}$, giving condition 1). Condition 2) is a consequence of the fact that the map $\pi_0 BG \to \pi_0 BH$ is an isomorphism and hence an epimorphism of sheaves under the assumption that $f_* : BG \to BH$ is a weak equivalence.

For the converse, it is evident that all maps

$$G(x,x) \to H(fx,fx)$$

are isomorphisms under the assumption that the first condition is satisfied, so that f induces an isomorphism in all fundamental groups for all local choices of base point. The second condition implies that the induced map $\pi_0 BG \to \pi_0 BH$ is an epimorphism of sheaves. To see that this map is a monomorphism, it suffices to show that the underlying presheaf map $\pi_0^p BG \to \pi_0^p BH$ is a monomorphism of presheaves, but this is a consequence of the assumption that all maps

$$G(x,y)(U) \to H(fx,fy)(U)$$

are isomorphisms. All sheaves of higher homotopy groups for BG and BH are trivial. □

2 Stacks

Suppose that $\mathcal{C}$ is a small Grothendieck site. A *stack* is normally defined to be a sheaf of groupoids G on $\mathcal{C}$ which satisfies the effective descent condition.

The effective descent condition (Definition 2 below) can be a bit of a mouthful — three equivalent descriptions are given here. Suppose that $R \subset \hom(\ ,U)$ is a covering sieve, and use the notation R for the full subcategory of $\mathcal{C} \downarrow U$ whose objects are the morphisms $\phi : V \to U$ of the sieve. In this notation, there

is a canonical functor $R \to \mathcal{C}$ which is defined by sending $\phi : V \to U$ to the object V, and $G|_R$ will denote the restriction of a sheaf or presheaf along this functor.

Following Giraud [4], an effective descent datum $x = \{x_\phi\} : R \to G$ on R with coefficients in G consists of objects $x_\phi \in G(V)$ for each $\phi : V \to U$ in R, and isomorphisms $c_\psi : \psi^* x_\phi \to x_{\phi\psi}$ for each composable pair

$$W \xrightarrow{\psi} V \xrightarrow{\phi} U$$

in R, such that $c_1 = 1$ and for each sequence

$$W' \xrightarrow{\omega} W \xrightarrow{\psi} V \xrightarrow{\phi} U,$$

the diagram

$$\begin{tikzcd}
\omega^*\psi^* x_\phi \arrow[r, "\omega^* c_\psi"] \arrow[d, equal] & \omega^* x_{\phi\psi} \arrow[d, "c_\omega"] \\
(\psi\omega)^* x_\phi \arrow[r, "c_{\psi\omega}"'] & x_{\phi\psi\omega}
\end{tikzcd}$$

commutes.

Alternatively, an effective descent datum can be viewed as a pseudo-natural transformation $R \to G|_R$, where R is identified with a functor taking values in discrete categories, and so a morphism of descent data is most naturally a homotopy of pseudo-natural transformations. This means that a morphism $f : x \to y$ of descent data on R consists of morphisms $f_\phi : x_\phi \to y_\phi$ in $G(V)$, for each $\phi : V \to U$ in R such that the diagrams

$$\begin{tikzcd}
\psi^* x_\phi \arrow[r, "\psi^* f_\phi"] \arrow[d, "c_\psi"'] & \psi^* y_\phi \arrow[d, "c_\psi"] \\
x_{\phi\psi} \arrow[r, "f_{\phi\psi}"'] & y_{\phi\psi}
\end{tikzcd}$$

Note that the collection of morphisms $x \to y$ of effective descent data $x, y : \hom(\ ,U) \to G$ can be identified up to isomorphism with the collection of morphisms $x_U \to y_U$ in $G(U)$.

Finally, there is a homotopy theoretic description of effective descent data which is quite persuasive even though it depends on the smallness of the underlying site $\mathcal{C}$. Observe that every covering sieve $R \subset \hom(\ ,U)$ determines a functor $R \to \mathbf{Shv}(\mathcal{C})$ which is defined by sending an object $\phi : V \to U$ of R to (the sheaf associated to) $V = \hom(\ ,V)$. The associated translation category E_R is best described in terms of its corresponding classifying object BE_R, which has sheaf of n-simplices given by the disjoint union

$$\bigsqcup_{\phi_0 \to \cdots \to \phi_n} V_0$$

indexed over strings of morphisms in R between objects $\phi_i : V_i \to U$. In other words, BE_R is the homotopy colimit of the objects appearing in the covering sieve R. From this point of view, an effective descent datum is a functor $E_R \to G$, or equivalently a simplicial sheaf map $BE_R \to BG$, and a morphism of effective descent data is a natural transformation $E_R \times \mathbf{1} \to G$, or a simplicial homotopy $BE_R \times \Delta^1 \to BG$.

Write $\hom(E_R, G)$ for the groupoid of effective descent data on R with coefficients in G. Observe that the associated nerve $B\hom(E_R, G)$ is isomorphic to the simplicial function space $\mathbf{hom}(BE_R, BG)$. This function space is a Kan complex, since classifying spaces of groupoids are Kan complexes.

Let E_U denote the translation category associated to the covering sieve $\hom(\ ,U)$ which is generated by the identity map 1_U on U. The set of path components $\pi_0 \hom(E_U, G)$ can be identified with the set $\pi_0 G(U)$ of path components of the groupoid $G(U)$, for any groupoid G. In effect, the functor $G(U) \to \hom(E_U, G)$ which is defined by $x \mapsto \{\phi^* x\}$ is an equivalence of categories.

Definition 2. A sheaf of groupoids G satisfies the *effective descent condition* if the restriction functor

$$\hom(E_U, G) \to \hom(E_R, G)$$

is an equivalence of categories for every covering seive $R \subset \hom(U,\ )$, and all objects $U \in \mathcal{C}$.

This definition is equivalent to the classical statement, which is the requirement that every effective descent datum on a covering sieve $R \subset \hom(\ ,U)$ taking values in G is determined up to isomorphism by some object of $G(U)$. Note that the composite functor

$$G(U) \to \hom(E_U, G) \to \hom(E_R, G)$$

is already fully faithful, since G is a sheaf. The effective descent condition is therefore equivalent to the requirement that

$$\pi_0 \hom(E_U, G) \to \pi_0 \hom(E_R, G)$$

is a bijection, or that the function

$$\pi_0 G(U) \to \pi_0 \hom(E_R, G)$$

is bijective.

The set of simplicial homotopy classes of maps $\pi(BE_R, BG)$ coincides with the set $\pi_0(E_R, G)$ of functors mod natural isomorphism. Thus, yet another way of stating the effective descent condition for G is to require that every simplicial map $BE_R \to BG$ extends to a map $BE_U \to BG$ which is unique up to naive homotopy.

Suppose that $\{\phi_i : V_i \to U\}$ is a covering family for the object U of $\mathcal{C}$, and that G is a stack. Write $V_\bullet$ for the Čech resolution of U which is associated to

the covering. A simplicial sheaf map $\alpha : V_\bullet \to BG$ consists of objects $\alpha_i \in G(V_i)$ and morphisms

$$\alpha_{i,j} : \alpha_i|_{V_i \times_U V_j} \to \alpha_j|_{V_i \times_U V_j}$$

of $G(V_i \times_U V_j)$, which satisfy the obvious cocycle condition (ie. composition law) in $G(V_i \times_U V_j \times_U V_k)$. We also require that the morphisms $\alpha_{i,i}$ are identities. Write $\hom(V_\bullet, BG)$ for the groupoid of cocyles and isomorphisms of cocycles. The Čech resolution $V_\bullet$ is the nerve of a trivial groupoid, so that the nerve $B\hom(V_\bullet, BG)$ coincides with the function space $\mathbf{hom}(V_\bullet, BG)$, and this function space is a Kan complex.

Suppose that R is the covering sieve associated to the family $\{V_i \to U\}$. Then R consists of those maps $\phi : V \to U$ which factor through some V_i. It is plain that any simplicial sheaf map $BE_R \to BG$ (for an arbitrary sheaf of groupoids G) restricts to a cocycle $V_\bullet \to BG$: the elements $x_i \in G(U_i)$ are defined by $x_i = x_{\phi_i}$, and the map $\alpha_{i,j}$ is defined by the composite isomorphism

$$pr_i^* x_i \to x_{p_{i,j}} \leftarrow pr_j^* x_j$$

arising from the diagram

$$\begin{tikzcd} V_i \times_U V_j \arrow[r, "pr_j"] \arrow[d, "pr_i"'] \arrow[dr, "p_{i,j}"] & V_j \arrow[d, "\phi_j"] \\ V_i \arrow[r, "\phi_i"'] & U \end{tikzcd}$$

Restriction plainly preserves isomorphism of the data, so we get a functor

$$\hom(E_R, G) \to \hom(V_\bullet, BG)$$

and hence a function

$$\pi(BE_R, BG) \to \pi(V_\bullet, BG)$$

relating naive homotopy classes.

Lemma 3. *Suppose that R is the covering sieve which is generated by a covering family $\{\phi_i : V_i \to U\}$. Then the restriction function $\pi(BE_R, BG) \to \pi(V_\bullet, BG)$ is a bijection for any sheaf of groupoids G.*

Proof. Suppose that the cocycle $\alpha : V_\bullet \to BG$ is defined by objects $\alpha_i \in G(V_i)$ and isomorphisms $\alpha_{i,j}$ in $G(V_i \times_U V_j)$. Choose specific factorizations

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[dr, "\phi"'] & V_i \arrow[d, "\phi_i"] \\ & U \end{tikzcd}$$

for all $\phi : V \to U$ in R, where ψ is the identity map if $\phi = \phi_i$ for some i. Define an object $x_\phi \in G(V)$ by $x_\phi = \psi^* \alpha_i$ for each ϕ. If $\tau : W \to V$ is any

other morphism, then the isomorphisms $\alpha_{i,j}$ induce a morphism $\tau^* x_\phi \to x_{\phi\tau}$ in $G(W)$, and the collection of all such elements and morphisms determine an effective descent datum $\{x_\phi\} : R \to G$ on account of the cocycle conditions for α. The assignment $\{x_i\} \mapsto \{x_\phi\}$ defines a functor $\hom(V_\bullet, BG) \to \hom(E_R, G)$ which is inverse to the restriction functor up to isomorphism. $\square$

We can now prove the following:

Proposition 4. *Suppose that G is a stack, and that U is an object of $\mathcal{C}$. Then there is an isomorphism*

$$[U, BG] \cong \pi_0 G(U).$$

Proof. There is an isomorphism

$$[U, BG] \cong \varinjlim_{V} \pi(V, BG),$$

where the colimit is indexed over hypercovers $V \to K(U, 0)$ of the object U. We can assume [6] that these hypercovers are of the form $V_\bullet \to U$ for some sheaf epimorphism $V \to U$, and then a cofinality argument allows us to presume that the sheaf epi $\bigsqcup V_i \to U$ arises from some covering family $\{V_i \to U\}$. The canonical map

$$\pi_0 G(U) \to \pi(V_\bullet, BG)$$

is the composite

$$\pi_0 G(U) \cong \pi_0 \hom(E_U, G) \to \pi_0 \hom(E_R, G) \to \pi_0 \hom(V_\bullet, BG)$$

if R is the sieve which is generated by the covering family $\{V_i \to U\}$. This composite is a bijection by the assumption that G is a stack, along with Lemma 3 $\square$

Recall that a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is said to be a *pointwise weak equivalence* if f induces a weak equivalence $f : X(U) \to Y(U)$ of simplicial sets for all objects $U \in \mathcal{C}$.

Lemma 5. *Suppose that G is a sheaf of groupoids and that x is a global section of* $\mathrm{Ob}(G)$. *Suppose that $BG \to X$ is a globally fibrant model of BG. Then the induced map $\Omega_x BG \to \Omega_x X$ is a pointwise weak equivalence.*

Proof. The section x is a global choice of basepoint $x : * \to BG$ for BG, and $\Omega_x BG$ is the corresponding loop object. Let the presheaf of groupoids G_x be the connected component of x in G in all sections, and observe that the inclusion $G(x, x) \to G_x$ is an equivalence of groupoids in all sections. Then there is an isomorphism $\Omega_x BG \cong \Omega_x BG_x$, and there are pointwise weak equivalences of simplicial presheaves

$$\Omega_x BG_x \leftarrow \Omega BG(x, x) \to PBG(x, x) \times_{BG(x,x)} EG(x, x) \leftarrow G(x, x)$$

The "simplicial" object $G(x, x)$ is a sheaf concentrated in degree 0 and is therefore globally fibrant. It follows that every local weak equivalence $\Omega_x BG \to Y$ with Y globally fibrant must be a pointwise weak equivalence. The map $\Omega_x BG \to \Omega_x X$ is such a map. $\square$

The following is the main result of this section. In colloquial terms, it asserts that classifying spaces of stacks satisfy descent.

Theorem 6. *Suppose that G is a stack of groupoids on a small site $\mathcal{C}$, and that $f : BG \to X$ is a globally fibrant model of its associated classifying object. Then f is a pointwise weak equivalence.*

Proof. Consider the restricted map $BG|_U \to X|_U$ on the site $\mathcal{C} \downarrow U$. This map is a local weak equivalence, and $X|_U$ is globally fibrant. The induced map in global sections is $BG(U) \to X(U)$. To show that this map is a weak equivalence of simplicial sets, it suffices to show that it induces an isomorphism $\pi_0 BG(U) \cong \pi_0 X(U)$ and to show that for any choice of base point $x \in BG(U)$ the induced map $\Omega_x BG(U) \to \Omega_x X(U)$ is a weak equivalence of simplicial sets (note that $X(U)$ is already a Kan complex, so this makes sense). The statement about path components is a consequence of Proposition 4, while the statement about loop spaces is a consequence of Lemma 5. □

3 Stack completion

Suppose that R is a covering sieve for U and that G is a sheaf of groupoids. Recall that the groupoid $\hom(E_R, G)$ has as objects all effective descent data $x = \{x_\phi\}$ for R, and all morphisms (or isomorphisms) $x \to y$ of descent data, defined in the obvious way. Every refinement $S \subset R$ of covering sieves determines a functor $\hom(E_R, G) \to \hom(E_S, G)$ defined by restriction $x \mapsto x|_S$ of descent data. Write

$$\mathrm{St}^p(G)(U) = \varinjlim_R \hom(E_R, G)$$

for the filtered colimit in the category of groupoids of all such categories of effective descent data. Every map $\phi : V \to U$ in the site $\mathcal{C}$ and every covering sieve R of U together determine a functor

$$\phi^* : \hom(E_R, G) \to \hom(E_{\phi^{-1}R}, G),$$

again by restriction of descent data. This functor respects the restriction functors coming from an inclusion $S \subset R$ of covering sieves, and so $\phi : V \to U$ induces a functor $\phi^* : \mathrm{St}^p(G)(U) \to \mathrm{St}^p(G)(V)$. Write $\mathrm{St}(G)$ for the sheaf of groupoids associated to the presheaf $\mathrm{St}^p(G)$. There is a natural morphism

$$\eta : G \to \mathrm{St}(G)$$

of sheaves of groupoids.

The object $\mathrm{St}(G)$ is the *stack completion* of G (or "associated stack" in [4]), but we have to show that the name makes sense. We shall prove that

1) the induced map $\eta_* : BG \to B\,\mathrm{St}(G)$ is a local weak equivalence of simplicial sheaves, and

2) the sheaf of groupoids $\mathrm{St}(G)$ is a stack.

Lemma 7. *Suppose that G is a sheaf of groupoids. Then the induced map $\eta_* : BG \to B\,\mathrm{St}(G)$ is a local weak equivalence of simplicial sheaves.*

Proof. We verify the conditions 1) and 2') for η — see Lemma 1.

We have already seen that the functor $G(U) \to \hom(E_R, G)$ is fully faithful, and η is a filtered colimit of such functors so that η is fully faithful, giving condition 1).

Suppose that $x = \{x_\phi\}$ is an object of $\hom(E_R, G)$, and suppose that $\phi : V \to U$ is a morphism of R. Then the covering sieve

$$\phi^{-1}(R) = \{\alpha : W \to V \mid \phi \cdot \alpha \in R\}$$

coincides with $\hom(\ ,V)$. It follows that the class $\phi^*(x)$ lifts to $G(V)$ up to isomorphism for all $\phi : V \to U$ in R. Every element of $\mathrm{Ob}(St(G))(U)$ lifts locally to $\mathrm{Ob}(St^p G)$, and we have just seen that every element of $\mathrm{Ob}(St^p G)(V)$ lifts locally up to isomorphism to an element of $\mathrm{Ob}(G)$. It follows that every element of $\mathrm{Ob}(StG)(U)$ lifts locally up to isomorphism to an element of $\mathrm{Ob}(G)$, and condition 2') is verified. $\square$

Lemma 8. *Suppose that G is a sheaf of groupoids, and suppose that $S \subset R \subset \hom(\ ,U)$ are covering sieves of U. Then the restriction functor*

$$\hom(E_R, G) \to \hom(E_S, G)$$

is fully faithful.

Proof. Suppose that $f : x|_S \to y|_S$ is a morphism of restricted descent data. Suppose that $\phi : V \to U$ is a morphism of R, and let $\psi : W \to V$ be a morphism of the covering sieve $\phi^{-1}S$ for V. Then there are morphisms $F_\psi : \psi^* x_\phi \to \psi^* y_\phi$ which are uniquely determined by the commutativity of the diagram

$$\begin{tikzcd} \psi^* x_\phi \arrow[r, "F_\psi"] \arrow[d, "c_\psi"'] & \psi^* y_\phi \arrow[d, "c_\psi"] \\ x_{\phi\psi} \arrow[r, "f_{\phi\psi}"'] & y_{\phi\psi} \end{tikzcd}$$

The maps F_ψ are compatible with composition in $\phi^{-1}S$, and therefore uniquely determine a map $F_\phi : x_\phi \to y_\phi$.

Seeing that the diagrams

$$\begin{tikzcd} \gamma^* x_\phi \arrow[r, "\gamma^* F_\phi"] \arrow[d, "c_\gamma"'] & \gamma^* y_\phi \arrow[d, "c_\gamma"] \\ x_{\phi\gamma} \arrow[r, "F_{\phi\gamma}"'] & y_{\phi\gamma} \end{tikzcd}$$

commute is a matter of applying the same principle that creates the maps F_ϕ: one shows that the diagram commutes after applying ω^*, where ω is any map

such that $\phi\gamma\omega$ is a member of the sieve S, and this is a consequence of the cocycle condition for x and y together with the definition of the morphisms F_ϕ.

We have shown that the restriction map

$$\hom(E_R, G)(x, y) \to \hom(E_S, G)(x|_S, y|_S)$$

is surjective. The maps $f_\phi : x_\phi \to y_\phi$ are uniquely determined by the restriction to S, by the same argument, so the restriction map is also a monomorphism. □

Lemma 9. *Suppose that G is a sheaf of groupoids. Then the sheaf of groupoids* $\mathrm{St}(G)$ *is a stack.*

Proof. We have to show that the functor $\mathrm{St}(G)(U) \to \hom(E_R, \mathrm{St}(G))$ is an equivalence of categories, for every $U \in \mathcal{C}$ and for every covering sieve $R \subset \hom(\ , U)$. We already know that the functor

$$H(U) \to \hom(E_R, H)$$

is fully faithful for arbitrary sheaves of groupoids H. It suffices to show that every object of $\hom(E_R, \mathrm{St}(G))$ lifts to $\mathrm{St}(G)(U)$ up to isomorphism.

Suppose that $x = \{x_\phi\}$ is an object of $\hom(E_R, \mathrm{St}(G))$. Then there is a refinement $S \subset R$ such that the restriction of x to S comes equipped with isomorphisms $d_\psi : \eta(z_\psi) \to x_\psi$ for each $\psi : W \to U$ in S — this follows from Lemma 7. The functor $\eta : G \to \mathrm{St}(G)$ is fully faithful, so that a unique descent datum structure in G for the sieve S is induced on the elements z_ψ by the descent datum structure on $x|_S$. In effect, given the string

$$W \xrightarrow{\omega} V \xrightarrow{\psi} U$$

the cocycle map $c_\omega : \omega^* z_\psi \to x_{\psi\omega}$ is the unique morphism which makes the following diagram of isomorphisms commute:

$$\begin{tikzcd}
\eta(\omega^* z_\psi) \arrow[r, equal] \arrow[d, "\eta(c_\omega)"'] & \omega^*\eta(z_\psi) \arrow[r, "\omega^* d_\psi"] & \omega^* x_\psi \arrow[d, "c_\omega"] \\
\eta(z_{\psi\omega}) \arrow[rr, "d_{\psi\omega}"'] & & x_{\psi\omega}
\end{tikzcd}$$

The diagram of canonical functors

$$\begin{tikzcd}
G(U) \arrow[r, "\eta"] \arrow[d] & \mathrm{St}(G)(U) \arrow[d] \\
\hom(E_S, G) \arrow[r, "\eta_*"'] \arrow[ur] & \hom(E_S, \mathrm{St}(G))
\end{tikzcd} \tag{3}$$

commutes, and we have just seen that $x|_S$ is in the image of

$$\eta_* : \hom(E_S, G) \to \hom(E_S, \mathrm{St}(G))$$

up to isomorphism. It follows that $x|_S$ lifts to an element $y \in \mathrm{St}(G)(U)$ up to isomorphism. Finally $y|_R$ and x restrict to isomorphic elements on S, so that there is an isomorphism $y|_R \to x$ by Lemma 8. $\square$

Lemma 10. *Suppose that G is a sheaf of groupoids. Then the simplicial presheaf $B\,\mathrm{St}^p(G)$ satisfies descent, in the sense that any globally fibrant model $B\,\mathrm{St}^p(G) \to X$ is a pointwise weak equivalence.*

Proof. It suffices to show that the map $\eta_* : \pi_0\,\mathrm{St}^p(G)(U) \to \pi_0\,\mathrm{St}(G)(U)$ which is induced by the associated sheaf map is a bijection, and that all presheaves of homomorphisms of $\mathrm{St}^p(G)$ are sheaves. These two conditions imply that $\eta : B\,\mathrm{St}^p(G) \to B\,\mathrm{St}(G)$ is a pointwise weak equivalence, and then one can invoke Theorem 6.

There is a commutative diagram of canonical functors

$$\begin{tikzcd} G(U) \arrow[r] \arrow[d] & \mathrm{St}^p(G)(U) \arrow[r] \arrow[d] & \mathrm{St}(G)(U) \arrow[d] \\ \hom(E_R, G) \arrow[r] \arrow[ur] & \hom(E_R, \mathrm{St}^p(G)) \arrow[r] & \hom(E_R, \mathrm{St}(G)) \end{tikzcd}$$

for any covering sieve R of U (this diagram refines (3)). For any object $x \in \mathrm{St}(G)(U)$ there is a covering sieve R such that the corresponding descent datum $\{\phi^* x\}$ is isomorphic to a descent datum $\{\eta(z_\phi)\}$ where $\{z_\phi\}$ is a descent datum in G. But then there is an element $y \in \mathrm{St}^p(G)(U)$ such that $\eta(y)$ and x restrict to isomorphic elements in $\hom(E_R, \mathrm{St}(G))$. It follows that there is an isomorphism $\eta(y) \cong x$. If two elements $x, y \in G(U)$ become isomorphic in $\mathrm{St}^p(G)(U)$ then there is an isomorphism $x|_R \to y|_R$ in $\hom(E_R, G)$ for some covering sieve R of U. The functor $G(U) \to \hom(E_R, G)$ is fully faithful, so that $x \cong y$ in $G(U)$ since G is a sheaf. It follows that the map

$$\eta_* : \pi_0\,\mathrm{St}^p(G)(U) \to \pi_0\,\mathrm{St}(G)(U)$$

is a bijection.

For the sheaf property, we start with a preliminary case. Suppose that $x, y \in \hom(E_R, G)$, and that there are maps $\alpha_\phi : x|_{\phi^{-1}R} \to y|_{\phi^{-1}R}$ for all $\phi \in R$. Then for all $\phi : V \to U$ in R, $\phi^{-1}R = \{1_V\}$, so that α_ϕ uniquely determines an isomorphism $\alpha_\phi : x_\phi \to y_\phi \in G(V)$. Any $\gamma : W \to V$, determines a commutative diagram

$$\begin{tikzcd} \gamma^* x_\psi \arrow[r, "\gamma^*\alpha_\psi"] \arrow[d, "c"'] & \gamma^* y_\psi \arrow[d, "c"] \\ x_{\psi\gamma} \arrow[r, "\alpha_{\psi\gamma}"'] & y_{\psi\gamma} \end{tikzcd}$$

where the cocycle structures come from R, since the maps α_ψ are compatible with restriction. The family $\{\alpha_\phi\}$ therefore determines a unique map $x \to y$ in $\hom(E_R, G)$.

Suppose that S is a covering sieve of U, and that there are maps $\alpha_\psi : x|_{\psi^{-1}R} \to y|_{\psi^{-1}R}$ for $\psi \in S$, and that these maps are compatible with restriction. Then for all $\psi : V \to U$ in the covering sieve $R \cap S$, the compatible family of morphisms α_ψ uniquely determines a morphism $x|_{R\cap S} \to y|_{R\cap S}$ of $\hom(E_{R\cap S}, G)$. If $\phi \in S$ and $\gamma : W \to V \in \phi^{-1}R$, then $\phi\gamma \in R \cap S$ and there is a commutative diagram of functors

$$\begin{tikzcd}
\hom(E_R, G) \arrow[r, "res"] \arrow[d, "\phi^*"'] & \hom(E_{R\cap S}, G) \arrow[d, "(\phi\gamma)^*"] \\
\hom(E_{\phi^1 R}, G) \arrow[r, "\gamma^*"'] & \hom(E_W, G)
\end{tikzcd}$$

The restriction functor is fully faithful by Lemma 8, so that there is a unique morphism $\beta : x \to y$ such that $\beta_{R\cap S} = \alpha$. But then

$$\gamma^*\phi^*(\beta) = (\phi\gamma)^*(\alpha) = \alpha_{\phi\gamma} = \gamma^*\alpha_\phi$$

for all γ, so that $\phi^*(\beta) = \alpha_\phi$ for all $\phi \in S$. The morphism β is already determined by its restriction to $R \cap S$, so there is a unique $\beta : x \to y$ in $\hom(E_R, G)$ such that $\phi^*(\beta) = \alpha_\phi$ for all $\phi \in S$.

More generally, if we have $x, y \in \hom(E_R, G)$ and a compatible family of morphisms $\omega_\phi : x_\phi \to y_\phi$ in $\mathrm{St}^p(G)(V)$ for all $\phi : V \to U$ in the covering sieve S, then in the diagram

$$\begin{tikzcd}
\hom(E_R, G) \arrow[r] \arrow[d, "\phi^*"'] & \mathrm{St}^p(G)(U) \arrow[d, "\phi^*"] \\
\hom(E_{\phi^{-1}R}, G) \arrow[r] & \mathrm{St}^p(G)(V)
\end{tikzcd}$$

the canonical functor $\hom(E_{\phi^{-1}R}, G) \to \mathrm{St}^p(G)(V)$ is fully faithful by Lemma 8, so that there is a unique morphism α_ϕ of $\hom(E_{\phi^{-1}R}, G)$ which maps to ω_ϕ, and the family $\{\alpha_\phi\}$ is compatible. It therefore follows from the previous case that the presheaf of morphisms $\mathrm{St}^p(G)(x, y)$ on $\mathcal{C} \downarrow U$ is a sheaf. $\square$

Joyal and Tierney show [11] that there is a closed model structure on the category of sheaves of groupoids on a small Grothendieck site $\mathcal{C}$ for which the cofibrations are the functors $G \to H$ that are injective on objects, and the weak equivalences are the internal equivalences. They say that a sheaf of groupoids G is a *strong stack* if it is fibrant for this closed model structure, and a *strong stack completion* $H \to H^\wedge$ for a sheaf of groupoids H is a choice of fibrant model within their closed model structure.

Observe that if $f : X \to Y$ is a local weak equivalence of simplicial sheaves on $\mathcal{C}$, then the associated morphism $f_* : \pi(X) \to \pi(Y)$ of sheaves of fundamental groupoids is an internal equivalence. Also, if f is a cofibration of simplicial presheaves, then the map $f_* : \pi(X) \to \pi(Y)$ is injective on objects, and hence

is a cofibration of sheaves of groupoids in the sense of Joyal and Tierney. The fundamental groupoid construction is left adjoint to the classifying space construction, so the simplicial sheaf BG is globally fibrant if G is a strong stack. It follows that any strong stack completion $H \to H^{\wedge}$ induces a globally fibrant model $BH \to BH^{\wedge}$ of the classifying simplicial sheaf BH of H. The strong stack completion $H^{\wedge}$ is itself a stack, since we have the following:

Lemma 11. *Any covering sieve $R \subset \hom(\ ,U)$ of an object U induces a local weak equivalence $BE_R \to K(U,0)$ of simplicial presheaves.*

The lemma implies that the induced map

$$BH^{\wedge}(U) \cong \mathbf{hom}(K(U,0), BH^{\wedge}) \to \mathbf{hom}(BE_R, BH^{\wedge})$$

is a weak equivalence of simplicial sets since $BH^{\wedge}$ is globally fibrant.

Proof of Lemma 11. Suppose that $W \in \mathcal{C}$, and consider the induced map of W-sections

$$\bigsqcup_{\phi_0 \to \cdots \to \phi_n} \hom(W, V_0) \to \hom(W, U).$$

The fibre F_ϕ over a fixed morphism $\phi : W \to U$ is the nerve of the category of factorizations

$$\begin{tikzcd} & V \arrow[d, "\psi"] \\ W \arrow[ur] \arrow[r, "\phi"'] & U \end{tikzcd}$$

of ϕ with $\psi \in R$. If $\phi : W \to U$ is a member of R and then this category is non-empty and has an initial object, namely the picture

$$\begin{tikzcd} & W \arrow[d, "\phi"] \\ W \arrow[ur, "1"] \arrow[r, "\phi"'] & U \end{tikzcd}$$

Find a factorization of the canonical map

$$\begin{tikzcd} BE_R \arrow[r] \arrow[d, "i"'] & K(U,0) \\ F \arrow[ur, "\pi"'] & \end{tikzcd}$$

where i is a pointwise weak equivalence and π is a pointwise Kan fibration. In W-sections, and in the case where the fibre F_ϕ of the canonical map over

$\phi : W \to U$ is non-empty, then the fibre of π over ϕ is a fibrant model of F_ϕ, and if ϕ is in R this space is contractible. It follows that all lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & F(W) \arrow[d, "\pi"] \\ \Delta^n \arrow[r] & K(U,0)(W) \end{tikzcd}$$

have solutions if $n > 0$. But also every vertex $\phi : W \to U$ lifts locally to BE_R since $\phi^{-1}(R)$ is a covering sieve for W. It follows that the map $\pi : F \to K(U,0)$ is a local trivial fibration, and hence a local weak equivalence. $\square$

We now have the following corollaries of Theorem 6, Lemma 7 and Lemma 9

Corollary 12. *Suppose that G is a stack on the site $\mathcal{C}$. Then any strong stack completion $G \to G^\wedge$ induces a pointwise weak equivalence $BG \to BG^\wedge$. In other words, the groupoid $G(U)$ is equivalent to $G^\wedge(U)$ in each section.*

Corollary 13. *Suppose that G is a sheaf of groupoids on $\mathcal{C}$. Then the stack completion* $\mathrm{St}(G)$ *is pointwise equivalent to any strong stack completion $G^\wedge$ of G.*

We're henceforth going to take the most inclusive point of view, and say that a presheaf of groupoids G is a *stack* if the associated simplicial presheaf BG satisfies descent. A morphism $G \to H$ of presheaves of groupoids is an *internal equivalence* if the associated map $BG \to BH$ is a local weak equivalence of simplicial presheaves. Finally, a *stack completion* of a presheaf of groupoids H is an internal equivalence $H \to H'$ such that H' is a stack. Stack completions exist from several points of view: one is entitled to use the composites

$$H \xrightarrow{\eta} \tilde{H} \to \tilde{H}'$$

where η is the associated sheaf functor, and $\tilde{H}'$ can be any of the strong stack completion $\tilde{H}^\wedge$, the stack completion $\mathrm{St}(\tilde{H})$ or the "pre-stack completion" $\mathrm{St}^p(\tilde{H})$. Any two stack completions of a presheaf of groupoids H are pointwise equivalent by the standard tricks.

The cocycle approach which is implicit in the construction $H \mapsto \mathrm{St}^p(H)$ appears most often in geometric applications. If, for example, G is a sheaf of groups, then the usual argument relating cocyles and torsors specializes to the assertion that the groupoid $\mathrm{St}^p(G)(U)$ is equivalent to the groupoid of G-torsors $\mathrm{Tors}(U,G)$ over an object U in $\mathcal{C}$. The set of path components $\pi_0 \mathrm{Tors}(U,G)$ is the non-abelian cohomology object $H^1(U,G) = [U, BG]$.

4 Some classification results

Suppose that G is a sheaf of groups on $\mathcal{C}$, and recall that a left G-torsor is a sheaf P equipped with a free left G-action $G \times P \to P$, such that the sheaf P/G

is isomorphic to the terminal sheaf $*$, and such that there is a sheaf epi $U \to *$ with an associated G-equivariant isomorphism

$$G \times U \cong U \times P.$$

Recall that it suffices to take $U = \bigsqcup_i U_i$, where U_i is a set of representables, by standard nonsense. Furthermore, triviality of P over U in the sense of the isomorphism above amounts to the existence of a section $\sigma : U \to P$ of P, for then the displayed isomorphism is defined by the map

$$(g, u) \mapsto (u, g \cdot \sigma(u)).$$

In particular, the G-torsor P is trivial, or equivariantly isomorphic to G, if and only if it has a global section.

The non-abelian cohomology object $H^1(\mathcal{C}, G)$ is the set of isomorphism classes of G-torsors on the site $\mathcal{C}$, as usual.

It is shown in [8] that there is an identification

$$H^1(\mathcal{C}, G) \cong [*, BG], \tag{4}$$

where the thing on the right is morphisms in the homotopy category of simplicial sheaves (or presheaves) on the site $\mathcal{C}$. The proof of this result starts with the Verdier hypercovering characterization

$$[*, BG] \cong \varinjlim_{V} \pi(V, BG)$$

of morphisms in the homotopy category (BG is locally fibrant), where $\pi(\ ,\ )$ indicates simplicial homotopy classes of maps and the filtered colimit is indexed over the hypercovers (or local trivial fibrations) $V \to *$. Simplicial homotopy classes of maps $V \to BG$ can be formally identified with homotopy classes of functors $\pi(V) \to G$ of sheaves of groupoids, where $\pi(V)$ is the fundamental groupoid of V in the sheaf category, and then the heart of the demonstration of the identification (4) is to show that map $V \to \operatorname{cosk}_0 V_0$ is isomorphic to the canonical map $V \to B\pi(V)$, or that $\pi(V)$ is isomorphic to the trivial groupoid on the sheaf V_0. Homotopy classes of functors $\pi(V) \to G$ may then be identified with equivalence classes of cocycles $\operatorname{cosk}_0 V_0 \to BG$ for the covering $V_0 \to *$, and so there is an isomorphism

$$\varinjlim_{V} \pi(V, BG) \cong \varinjlim_{V_0} \pi(\operatorname{cosk}_0 V_0, BG).$$

Thus, $[*, BG]$ is identified with the Čech invariant $\check{H}^1(\mathcal{C}, G)$, which is well known to coincide with the set of isomorphism classes of G-torsors.

Suppose that I is a small category. A set-valued functor $X : I \to \mathbf{Sets}$ determines a simplicial set EX and a map $\pi : EX \to BI$ such that if $0 : \mathbf{0} \to \mathbf{n}$

is the ordinal number map which picks out the object 0 then the diagram

$$\begin{tikzcd} EX_n \arrow[r, "0^*"] \arrow[d, "\pi"'] & EX_0 \arrow[d, "\pi"] \\ BI_n \arrow[r, "0^*"'] & BI_0 \end{tikzcd}$$

is a pullback. In effect, EX is the homotopy colimit, with

$$EX_n = \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0),$$

and π is the obvious map.

Conversely, if $\pi : Y \to BI$ is a simplicial set map such that all diagrams

$$\begin{tikzcd} Y_n \arrow[r, "0^*"] \arrow[d, "\pi"'] & Y_0 \arrow[d, "\pi"] \\ BI_n \arrow[r, "0^*"'] & BI_0 \end{tikzcd} \tag{5}$$

are pullbacks, then $Y \cong EX$ for some functor $X : I \to \mathbf{Sets}$. To see this, set $X(i) = \pi^{-1}(i)$ for the map $Y_0 \to BI_0$. Also

$$Y_n = \bigsqcup_{\underline{i} = i_0 \to \cdots \to i_n} \pi^{-1}(\underline{i})$$

The pullback squares (5) induce isomorphisms $\pi^{-1}(\underline{i}) \cong \pi^{-1}(i_0) = X(i_0)$ of fibres, which together determine an isomorphism over BI of the simplicial set Y with a simplicial set having n-simplices of the form

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0)$$

such that all diagrams

$$\begin{tikzcd} X(i_0) \arrow[r, "1"] \arrow[d, "in_{\underline{i}}"'] & X(i_0) \arrow[d, "in_{i_0}"] \\ \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \arrow[r, "0^*"'] & \bigsqcup_i X(i) \end{tikzcd}$$

commute. The map

$$d_0 : \bigsqcup_{\alpha : i_0 \to i_1} X(i_0) \to \bigsqcup_i X(i)$$

defines functions $\alpha_* : X(i_0) \to X(i_1)$ on summands. Any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ such that $\theta(0) = 0$ necessarily induces a function

$$\theta^* : \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \to \bigsqcup_{j_0 \to \cdots \to j_m} X(j_0)$$

which restricts to identity maps on summands, by comparison of pullbacks. It follows that an arbitrary ordinal number map $\gamma : \mathbf{r} \to \mathbf{n}$ induces a function

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \to \bigsqcup_{j_0 \to \cdots \to j_r} X(j_0)$$

which is induced on summands by the maps $X(i_0) \to X(i_{\theta(0)})$ determined by the morphism $i_0 \to i_{\theta(0)}$ in the underlying index category I. The assignments of $\alpha_* : X(i) \to X(j)$ to the maps $\alpha : i \to j$ therefore determine a functor $X : I \to \mathbf{Sets}$ such that $EX \cong Y$.

It follows that the category of functors $X : I \to \mathbf{Sets}$ and natural transformations is equivalent to the category of simplicial set maps $\pi : Y \to BI$ satisfying the pullback condition (5), with fibrewise maps over BI as morphisms. The equivalence arises from the homotopy colimit construction.

This equivalence is natural, and therefore gives an internal description of sheaf-valued functors $I \to \mathbf{Shv}(\mathcal{C})$ defined on a small sheaf of categories I over a site $\mathcal{C}$.

Suppose now that G is a sheaf of groupoids on $\mathcal{C}$. A *G-torsor* on $\mathcal{C}$ can be defined to by a simplicial sheaf map $Y \to BG$ which satisfies the pullback condition (5), and such that the canonical map $Y \to *$ is a local weak equivalence. From the development above, Y is the nerve EX of a sheaf of groupoids, so that the condition that $Y \to *$ is equivalent to requiring that G acts transitively and effectively on the sheaf Y_0. If G is a sheaf of groups, then this definition says that a G-torsor is the G-space Y_0 and

$$Y \cong EG \times_G Y_0 \simeq *,$$

which is equivalent to the classical definition. The set of G-torsors and natural transformations (or deck transformations) between them forms a category which will be denoted by $\mathrm{Tors}(*, G)$.

The sheaf of path components $\pi_0 G$ of BG determines a local fibration $BG \to \pi_0 G$. Take a global section $x \in \pi_0 G(*)$. Then the fibre over x is the classifying object BG_x of a sheaf of groupoids G_x which we shall call the *path component* of x. Note that G_x is locally connected. If $Y \to BG$ is a G-torsor, then Y is locally connected so that it factors through a unique path component $BG_x \subset BG$ of G. If $\alpha : Y \to Y'$ is a morphism of G-torsors, then Y and Y' factor through the same path component BG_x. It follows that

$$\mathrm{Tors}(*, G) = \bigsqcup_{x \in \pi_0 G(*)} \mathrm{Tors}(*, G_x).$$

If G_x has a global section y, then the inclusion $G_x(y, y) \to G_x$ is an internal equivalence, and any map $\theta : Y \to Y'$ of G_x-torsors pulls back to a morphism θ_* of $G_x(y, y)$-torsors. This map θ^* of group-valued torsors must be an isomorphism, so that θ is an isomorphism as well since G_x is locally connected. It follows that every morphism of G-torsors is an isomorphism, for arbitrary sheaves of groupoids G.

Every G-torsor $Y \to BG$ has within it a hypercover $Y \to *$, and every morphism $Y \to Y'$ of torsors is a morphism of hypercovers. It follows that the map

$$\operatorname{Ob}\operatorname{Tors}(*, G) \to [*, BG]$$

defined by sending $Y \to BG$ to the composite $* \xleftarrow{\simeq} Y \to BG$ in the homotopy category factors through a function

$$\pi_0 \operatorname{Tors}(*, G) \to [*, BG]$$

Theorem 14. *The function $\pi_0 \operatorname{Tors}(*, G) \to [*, BG]$ is a natural bijection for all sheaves of groupoids G.*

Proof. All (sheaf theoretic) cocycles $V_\bullet \to BG$ and all G-torsors $Y \to BG$ are of the form $Bf : BI \to BG$ where $f : I \to G$ is a functor defined on a groupoid I which is trivial in the sense that $I(x, x) = *$ for all local choices of objects x and I is locally connected. Note that all such maps are necessarily cocycles.

Suppose given such a functor $f : I \to G$ and make the homotopy colimit construction

$$\begin{tikzcd}
BI(U) \arrow[d, "f"'] & \bigsqcup_{x_0 \to \cdots \to x_n} B(f \downarrow x_0) \arrow[l, "\alpha"', "\simeq"] \arrow[r, "\beta"] \arrow[d] & \bigsqcup_{x_0 \to \cdots \to x_n} \pi_0 B(f \downarrow x_0) \arrow[d, "h(f)"] \\
BG(U) & BG(U) \arrow[l, "1"] \arrow[r, "1"'] & BG(U)
\end{tikzcd}$$

on the presheaf level in each section. The map α is the standard weak equivalence associated to the functor $f : BI(U) \to BG(U)$, and the map β is induced by the simplicial set maps $B(f \downarrow x_0) \to \pi_0 B(f \downarrow x_0)$ — these maps are weak equivalences since all components of the groupoid $I(U)$ are trivial. The presheaf map $h(f)$ represents a G-torsor, since the total space has the right homotopy type. This map is canonically isomorphic to $f : BI \to BG$ if f is a G-torsor, or homotopy colimit of sheaves over G: in that case $I = EX$ for some functor X, and there is a natural isomorphism $\pi_0 B(f \downarrow x_0) \cong X(x_0)$.

Given a cocycle $f : V_\bullet \to BG$, the associated G-torsor $h(f)$ represents an element mapping to f via the function

$$\pi_0 \operatorname{Tors}(*, G) \to [*, BG].$$

If $f : Y \to BG$ is a G-torsor then $h(f) \cong f$ in the category $\operatorname{Tors}(*, G)$. Further, any homotopy $BI \times \Delta^1 \to BG$ extends to a map $BI \times BG(\mathbf{1}) \to BG$ where $G(\mathbf{1})$ is the trivial groupoid on two objects. If I is locally connected and trivial, then so is $I \times G(\mathbf{1})$. Thus, any homotopy $f \simeq f'$ of cocycles determines torsors $h(f)$ and $h(f')$ which are in the same component of $\operatorname{Tors}(*, G)$. It follows that G-torsors f and f' are in the same component of $\operatorname{Tors}(*, G)$ if $h(f)$ and $h(f')$ are homotopic up to refinement. $\square$

Remark 15. The homotopy colimit construction in the proof of Theorem 14 specializes to the standard construction of a G-torsor from a G-cocycle, in the case where G is a sheaf of groups.

The last major result of this section identifies the stack completion of a Borel construction.

Suppose that G is a sheaf of groups and that N is a sheaf on $\mathcal{C}$ carrying a G-action. A G-torsor over N consists of a G-torsor P, together with a G-equivariant map $p : P \to N$. There is a groupoid $\mathrm{Tors}(\mathcal{C}, G)/N$ of such things, whose objects are the G-torsors over N, and whose morphisms are the G-equivariant commutative diagrams

$$\begin{tikzcd} P \arrow[dr, "p"] \arrow[d, "\theta"', "\cong"] & \\ P' \arrow[r, "p'"'] & N \end{tikzcd}$$

The groupoid $\mathrm{Tors}(\mathcal{C}, G)/N$ is the groupoid of global sections of a stack

$$\mathbf{Tors}(\mathcal{C}, G)/N,$$

which is usually called the *quotient stack* for the action of G on the sheaf N.

Write $E_G N$ for the translation groupoid which is associated to the action of G on N. The sheaf of objects of the category $E_G N$ is N, and the sheaf of morphisms is the product $G \times N$, with composeability and composition of morphisms defined in the standard way. The classifying object $B(E_G N)$ coincides with the Borel construction $EG \times_G N$.

The path components of the groupoid $\mathrm{Tors}(\mathcal{C}, G)/N$ can be characterized up to isomorphism as follows:

Theorem 16. *Suppose that G is a sheaf of groups on a site $\mathcal{C}$, and that N is a sheaf carrying a G-action. Then there is a bijection*

$$\pi_0(\mathrm{Tors}(\mathcal{C}, G)/N) \cong [*, EG \times_G N].$$

Proof. Suppose that $U \to *$ is a sheaf epi, and let $T(U)$ be the associated trivial groupoid with objects U and morphisms $U \times U$. It suffices to show that the set $\pi(T(U), E_G N)$ of homotopy classes of functors $T(U) \to E_G N$ is isomorphic to the set of isomorphism classes of G-torsors $P \to N$ which trivialize over U.

If the object $p : P \to N$ trivializes over U, there is a section $\sigma : U \to P$, and the composite

$$U \xrightarrow{\sigma} P \xrightarrow{p} N$$

is the object level map for a functor $T(U) \to E_G N$; the map of morphisms is the map $U \times U \to G \times N$ which is defined in sections by the assignment $(u_1, u_2) \mapsto (\tau(u_1, u_2), p\sigma(u_1))$. The element $\tau(u_1, u_2)$ is the unique element such that $\tau(u_1, u_2)\sigma(u_1) = \sigma(u_2)$ — τ is, in other words, the usual cocycle defined by the section σ. If $\sigma' : U \to P$ is a second such section, then $\sigma(u) = h(u)\sigma'(u)$ for some unique element $h(u)$ of G, and the associated cocyles τ and τ' are conjugate by h in the usual way. The map $h : U \to G \times N$ given by $u \mapsto (h(u), p\sigma(u))$ defines a homotopy of the functors $T(U) \to E_G N$ associated

to σ and σ'. It follows that isomorphic objects of $\mathrm{Tors}(\mathcal{C}, G)/N$ which trivialize over U determine homotopic functors.

If $f : T(U) \to E_G N$ is a functor, then the composite

$$T(U) \xrightarrow{f} E_G N \to BG$$

determines a G-torsor P which trivializes over U in the usual way, via the G-equivariant coequalizer

$$G \times U \times U \rightrightarrows G \times U \to P,$$

where the parallel arrows are defined, respectively, by

$$\begin{aligned}(g, u_1, u_2) &\mapsto (g\tau(u_1, u_2), u_1) \text{ and}\\ (g, u_1, u_2) &\mapsto (g, u_2).\end{aligned}$$

Here τ is induced by the functor f in the sense that

$$f(u_1, u_2) = (\tau(u_1, u_2), f(u_1)).$$

Composing with the G-equivariant map $f_* : G \times U \to N$ defined by $(g, u) \mapsto g \cdot f(u)$ has the same effect on the parallel pair defining P, and so f_* factors through a unique G-equivariant map $p : P \to N$. If the functors

$$f, f' : T(U) \to E_G N$$

are homotopic via some map $h : U \to G \times N$, with $h(u) = (h_1(u), f(u))$, (so that h is a homotopy $f \to f'$) then the morphism $G \times U \times U \to G \times U \times U$ defined by

$$(g, u_1, u_2) \mapsto (gh_1(u_2), u_1, u_2)$$

and $h_* : G \times U \to G \times U$ defined by

$$(g, u) \mapsto (gh_1(u), u)$$

together determine a G-equivariant isomorphism $h_* : P' \to P$ on the coequalizer level. At the same time the G-equivariant diagram

$$\begin{tikzcd} G \times U \arrow[dr, "f'_*"] \arrow[dd, "h_*"'] & \\ & N \\ G \times U \arrow[ur, "f_*"'] & \end{tikzcd}$$

commutes, so that the induced diagram

$$\begin{tikzcd} P' \arrow[dr, "f'_*"] \arrow[dd, "h_*"'] & \\ & N \\ P \arrow[ur, "f_*"'] & \end{tikzcd}$$

commutes as well. $\square$

Suppose again that G is a sheaf of groups. Recall [2] that a G-bitorsor consists of a G-torsor P and a G-equivariant map $P \to \mathbf{Aut}(G)$, where G acts on the sheaf of automorphisms of G by conjugation. A morphism of G-bitorsors is the obvious thing, namely a commutative diagram

$$\begin{tikzcd} P \arrow[d, "f"'] \arrow[dr] & \\ & \mathbf{Aut}(G) \\ P' \arrow[ur] & \end{tikzcd}$$

where f is G-equivariant. More generally, given sheaves of groups G and H which are locally isomorphic, a (G,H)-bitorsor consists of a G-torsor P and a G-equivariant map $P \to \mathbf{Iso}(H,G)$ where G acts on the sheaf of isomorphisms $\mathbf{Iso}(H,G)$ by composition with conjugation. A morphism of (G,H)-torsors is a G-equivariant commutative diagram, by analogy with the above.

Corollary 17. *1) The set* $[*, EG \times_G \mathbf{Iso}(H,G)]$ *is isomorphic to the set of isomorphism classes of* (G,H)*-bitorsors.*

2) The set $[*, EG \times_G \mathbf{Aut}(G)]$ *is isomorphic to the set of isomorphism classes of* G*-bitorsors.*

Proof. Statement 2) is the case $G = H$ of statement 1), and 1) follows from Theorem 16. ∎

Example 18. Suppose that S is a Noetherian scheme which is of finite dimension, and is integral and normal. Let $Sch|_S$ denote the category of schemes which are of finite type over S.

Following [14], suppose that $\pi : X \to S$ is a finite surjective morphism such that X is integral, and let $f : Y \to S$ be the normalization of X in a normal extension of $k(S)$ which contains $k(X)$. Then the natural map

$$\hom_S(Y,X) \to \hom_{k(S)}(k(X), k(Y))$$

is bijective since Y is normal, and the map $f : Y \to S$ is a pseudo-Galois covering in the sense that $k(Y)/k(S)$ is normal and the natural homomorphism

$$Aut_S(Y) \to Gal(k(Y)/k(S))$$

is an isomorphism. Write G for the Galois group of the extension $k(Y)/k(S)$.

Suppose that F is a presheaf of spectra on $Sch|_S$ which is globally fibrant for some topology. There is a universal pairing

$$t_\pi : \pi_*(F|_X) \wedge (EG \times_G Y)_+ \to F$$

in the stable category of presheaves of spectra on $Sch|_S$, and global sections $\sigma : P \to Y$ of the quotient stack $\mathrm{St}(E_G Y)$ (if they exist — they may not) determine transfer maps

$$t_{\pi,\sigma} : F(S' \times_S X) \to F(S'), \tag{6}$$

Of course, the meaning of this statement varies with the topology on the site $Sch|_S$. This is really a *separable transfer*, in contrast with the transfer maps constructed in [14], as only the separable degree of the extension $k(X)/k(S)$ appears in calculations. This construction specializes to a well defined transfer for all finite étale maps π, for both the étale and the qfh topologies. The pairing (6) is the subject of [10].

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