Chapter 5
Chern classes in motivic cohomology

We now show how the classical construction of Chern classes for a Bloch-Ogus cohomology theory give Chern classes in motivic cohomology. For simplicity, we assume in this section that $\mathcal{V}$ is the full category $\text{Sm}_S$; this is not strictly necessary, but allows us to avoid specifying which vector bundles occur in the category $\mathcal{V}$.

5.1. Multiplication of cosimplicial objects

We describe how one gives a multiplicative structure to cosimplicial objects in certain symmetric monoidal categories.

(5.1.1) External products

We recall the standard construction of products for cosimplicial objects in a tensor category.

We have the category $\Delta$ (cf. (4.1.1)); recall the co-boundary maps (4.1.1.2)

$$\delta^i_n: [n] \to [n + 1]; \quad i = 0, \ldots, n + 1.$$ 

Let $A$ be an additive category, and let

$$X: \Delta \to A$$

be a cosimplicial object in $\mathcal{A}$. We may form the object $X^* = X^+(A)$:

$$X^* := X^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{n-1}} X^n \xrightarrow{d_n} \cdots$$

where $X^n = X([n])$, and $d_n: X^n \to X^{n+1}$ is the usual alternating sum

$$d_n = \sum_{i=0}^{n+1} (-1)^i X(\delta^n_i).$$

We may also form the various truncations of $X^*$:

$$X^m \leq \star \leq n := X^m \xrightarrow{d_m} \cdots \xrightarrow{d_{n-1}} X^n.$$ 

If $p$ and $q$ are positive integers, we have the maps

$$f_1^{p,q}: [p] \to [p + q]$$

$$f_2^{p,q}: [q] \to [p + q]$$

(5.1.1.1)
given by

\[ f_1^{p,q}(i) = i; \quad f_2^{p,q}(i) = i + p. \]

Suppose that \( \mathcal{A} \) is a tensor category with tensor product \( \otimes \). Let

\[ X: \Delta \to \mathcal{A}; \quad Y: \Delta \to \mathcal{A} \]

be cosimplicial objects of \( \mathcal{A} \), giving the diagonal cosimplicial object

\[ X \otimes Y: \Delta \to \mathcal{A}, \]

\[ (X \otimes Y)_n = X_n \otimes Y_n, \]

\[ (X \otimes Y)(g) = X(g) \otimes Y(g). \]

We have as well the tensor product double complex \( X^* \otimes Y^* \), and the complex \( (X \otimes Y)^* \).

Let

\[ \cup_{p,q}^n: X^p \otimes Y^q \to X^n \otimes Y^n \]

be the map

\[ \cup_{p,q}^n = X(f_1^{p,q}) \otimes Y(f_2^{p,q}), \]

and define

\[ \cup_{X,Y}^n: \bigoplus_{p+q=n} X^p \otimes Y^q \to X^n \otimes Y^n \]

by

\[ \cup_{X,Y}^n = \sum_{p+q=n} \cup_{p,q}^n. \]

The relation (of linear combinations of maps from \([p] \coprod [q] \) to \([ p + q + 1 ] \))

\[ \sum_{i=0}^{p+1} (-1)^i [(f_1^{p+1,q} \circ \delta_i^p) \coprod f_2^{p+1,q}] + \sum_{i=0}^{q+1} (-1)^i [f_1^{p,q+1} \coprod (f_2^{p+1,q} \circ \delta_i^q)] = \sum_{i=0}^{p+q+1} (-1)^i \delta_i^{p+q} \circ (f_1^{p,q} \coprod f_2^{p,q}) \]

implies that the maps (5.1.1.2) define the map of complexes

\[ \cup_{X,Y}: \text{Tot}(X^* \otimes Y^*) \to (X \otimes Y)^*. \]

One easily verifies that the maps \( \cup_{X,Y} \) are associative, in the obvious sense; it is well known that the maps \( \cup_{X,Y} \) are graded-commutative, up to functorial homotopy.
Multiplication in a symmetric monoidal category

Let \((A, \otimes, \tau, \mu, 1)\) be a symmetric monoidal category. We have the diagonal functor
\[
\Delta_A: A \to A \times A
\]
\[
\Delta_A(X) = (X, X)
\]
\[
\Delta_A(f: X \to Y) = (f, f).
\]

A commutative multiplication in \(A\) is a natural transformation
\[
m: \otimes \circ \Delta_A \to \text{id}_A
\]
such that
\[
m \circ (m \times \text{id}_A) = m \circ (\text{id}_A \times m)
\]
\[
m \circ (\tau \circ \Delta_A) = m
\]
\[
m(1) = \mu_1: 1 \otimes 1 \to 1.
\]

Let \(A_\mathbb{Z}\) denote the free additive category generated by \(A\). The symmetric monoidal structure on \(A\) induces the structure of a tensor category on \(A_\mathbb{Z}\).

Cup products

Suppose that \(A\) is a symmetric monoidal category with multiplication \(m\), and \(X\) is a cosimplicial object in \(A\).

Define the map of cosimplicial objects
\[
m_X: X \otimes X \to X
\]
by
\[
m_X([n]) = m_{X([n])}: X([n]) \otimes X([n]) \to X([n]);
\]
we let
\[
m_X^*: (X \otimes X)^* \to X^*
\]
be the map induced by \(m_X\). We may then define the map
\[
m(X^*): \text{Tot}(X^* \otimes X^*) \to X^*
\] (5.1.3.1)
by
\[
m(X^*) = m_X^* \circ \cup_{X, X}.
\]

For a complex \(X^*\), we \(X^{m \leq *} \leq n\) denote the truncation of \(X\) to degrees \(d, d \leq n\), and \(X^{m \leq * \leq n}\) the truncation to degrees \(d, m \leq d \leq n\). Taking truncations gives the maps
\[
m^n(X^{m \leq * \leq n}): \text{Tot}(X^{m \leq * \leq n} \otimes X^{* \leq n'}) \to X^{m \leq * \leq n}.
\] (5.1.3.2)
for all \( n' \geq n \). For \( m' \leq m \), and \( n \leq n' \leq n'' \), the diagrams
\[
\begin{align*}
\text{Tot}(X^{m \leq * \leq n} \otimes X^{* \leq n'}) & \xrightarrow{m''(X^{m \leq * \leq n})} X^{m \leq * \leq n} \\
\downarrow \quad & \downarrow \\
\text{Tot}(X^{m' \leq * \leq n} \otimes X^{* \leq n'}) & \xrightarrow{m''(X^{m' \leq * \leq n})} X^{m' \leq * \leq n}
\end{align*}
\]
and
\[
\begin{align*}
\text{Tot}(X^{m \leq * \leq n'} \otimes X^{* \leq n''}) & \xrightarrow{m''(X^{m \leq * \leq n'})} X^{m \leq * \leq n'} \\
\downarrow \quad & \downarrow \\
\text{Tot}(X^{m \leq * \leq n} \otimes X^{* \leq n''}) & \xrightarrow{m''(X^{m \leq * \leq n})} X^{m \leq * \leq n}
\end{align*}
\] commute, and for \( n \leq n' \leq n'' \), the diagram
\[
\begin{align*}
\text{Tot}(X^{m \leq * \leq n} \otimes X^{*}) & \xrightarrow{m(X^{m \leq * \leq n})} X^{m \leq * \leq n} \\
\| \quad & \| \\
\text{Tot}(X^{m \leq * \leq n} \otimes X^{* \leq n''}) & \xrightarrow{m''(X^{m \leq * \leq n})} X^{m \leq * \leq n} \\
\downarrow \quad & \downarrow \\
\text{Tot}(X^{m \leq * \leq n} \otimes X^{* \leq n'}) & \xrightarrow{m''(X^{m \leq * \leq n})} X^{m \leq * \leq n}
\end{align*}
\] commutes.

When the indices are obvious, we write
\[
\cup_X : \text{Tot}(X^{m \leq * \leq n} \otimes X^{* \leq n'}) \to X^{m \leq * \leq n}. \tag{5.1.3.6}
\]
for the map (5.1.3.2).

**Remark (5.1.4)** The maps (5.1.3.1) are associative. The maps \( m(X^{*}) \) are not in general commutative, but are commutative up to homotopy. Thus, suppose we have a graded tensor functor
\[
F : \mathbf{K}^b(A_{\mathbb{Z}}) \to \mathcal{B}.
\]
Then the maps \( F(m_n(X^{* \leq n})) \) give \( \text{Hom}_B(1_{\mathcal{B}}, F(X^{* \leq n})) \) the structure of a (possibly non-unital) graded-commutative ring, and \( \text{Hom}_B(1_{\mathcal{B}}, F(X^{m \leq * \leq n})) \) has the structure of a graded \( \text{Hom}_B(1_{\mathcal{B}}, F(X^{* \leq n})) \)-module via the maps \( F(m_n(X^{m \leq * \leq n})) \).
5.2. Motives of simplicial schemes

We describe objects of $\mathcal{DM}(\mathcal{V})$ associated to simplicial objects of $\mathcal{L}(\mathcal{V})$ and $\mathcal{V}$.

(5.2.1)

We have the full sub-category $\Delta^{\leq n}$ be of $\Delta$ with objects $[0], \ldots, [n]$, and the inclusions

$$t_n: \Delta^{\leq n}_{\op} \rightarrow \Delta^\op, \quad t_{n',n}: \Delta^{\leq n}_{\op} \rightarrow \Delta^{\leq n'}_{\op}; \quad n < n'. $$

If we have a functor 

$$(X,f): \Delta^{\leq n}_{\op} \rightarrow \mathcal{L}(\mathcal{V}),$$

we may compose $(X,f)^{\op}$ with the functor

$$Z(q): \mathcal{L}(\mathcal{V})^{\op} \rightarrow \mathcal{A}_{mot}(\mathcal{V}),$$

forming the functor

$$Z(q) \circ (X,f): \Delta^{\leq n} \rightarrow \mathcal{A}_{mot}(\mathcal{V}).$$

We let

$$Z_X(q)_f \in C^b_{mot}(\mathcal{V})$$

be the object of the category of complexes $C^b_{mot}(\mathcal{V})$ associated to the functor (5.2.1.1); for $0 \leq m \leq n$, we let $Z_X(q)_f^{m^{\leq s^{\leq n}}}$ be the associated truncation of $Z_X(q)_f$.

Sending $(X,f)$ to $Z_X(q)_f$ or $Z_X(q)_f^{m^{\leq s^{\leq n}}}$ defines the functors

$$Z_{(-)}(q)_{(-)}: \mathcal{L}(\mathcal{V})(\Delta^{\leq n}_{\op}) \rightarrow C^b_{mot}(\mathcal{V})$$

and

$$Z_{(-)}(q)_{(-)}^{m^{\leq s^{\leq n}}}: \mathcal{L}(\mathcal{V})(\Delta^{\leq n}_{\op}) \rightarrow C^b_{mot}(\mathcal{V});$$

we may also consider $Z_{(-)}(q)_{(-)}$ and $Z_{(-)}(q)_{(-)}^{m^{\leq s^{\leq n}}}$ as functors with values in $K^b_{mot}(\mathcal{V})$ or $D^b_{mot}(\mathcal{V})$ as the need arises. Clearly the functors (5.2.1.4) factors through the functor

$$\mathcal{L}(\mathcal{V})(\Delta^{\leq n}_{\op}) = \mathcal{L}(\mathcal{V})^{\op}(\Delta^{\leq n}) \rightarrow C^b(\mathcal{L}(\mathcal{V})^{\op})_Z$$

$$X \mapsto X^*,$$

via the extension of (5.2.1.1) to

$$Z(q): C^b(\mathcal{L}(\mathcal{V})^{\op}) \rightarrow C^b_{mot}(\mathcal{V}).$$

(5.2.2) Lifting simplicial objects to $\mathcal{L}(\mathcal{V})$
Let $X$ be a simplicial object in $\mathcal{V}$:

$$X: \Delta^{\text{op}} \to \mathcal{V}.$$ 

Fix an integer $n \geq 0$; for each $m \leq n$, we let $f_{m,n}$ be the map

$$f_{m,n} = \prod_{\{f:[m] \to [k], 0 \leq k \leq n\}} X(f) \prod_{\{f:[m] \to [k], 0 \leq k \leq n\}} X_k \to X_m.$$ 

The reader will easily verify that each map

$$X(g): X_m \to X_k; \quad g:[k] \to [m]$$

lifts to the map in $\mathcal{L}(\mathcal{V})$

$$X(g): (X_m, f_{m,n}) \to (X_k, f_{k,n}).$$

We let

$$(X, f_X): \Delta^{\leq \text{nop}} \to \mathcal{L}(\mathcal{V}) \tag{5.2.2.1}$$

be the functor lifting $X$ with

$$(X, f_X)_m = (X_m, f_{m,n}).$$

We let

$$\mathbb{Z}(q)(X): \Delta^{\leq n} \to \mathcal{A}_{\text{mot}}(\mathcal{V}) \tag{5.2.2.2}$$

be the composition of (5.2.2.1) with the functor (5.2.1.1), and let $\mathbb{Z}_X(q)^{m \leq * \leq n}$ be the truncated complex in $\mathcal{C}^b_{\text{mot}}(\mathcal{V})$ associated to (5.2.2.2).

Sending $X$ to $\mathbb{Z}_X(q)^{m \leq * \leq n}$ defines a functor

$$\mathbb{Z}_(-)(q)^{m \leq * \leq n}: \mathcal{V}^{\text{op}}(\Delta^{\leq n}) \to \mathcal{D}^b_{\text{mot}}(\mathcal{V}). \tag{5.2.2.3}$$

Indeed, given a map

$$p: Y \to X$$
in $\mathcal{V}(\Delta^{\leq \text{nop}})$, let $g_m: X'_m \to X_m$ be the map

$$g_m = \prod_{\{f:[m] \to [k], 0 \leq k \leq n\}} X(f) \cup p_m \circ Y(f) \prod_{\{f:[m] \to [k], 0 \leq k \leq n\}} X_k \prod_{\{f:[m] \to [k], 0 \leq k \leq n\}} Y_k \to X_m.$$ 

This then defines the lifting of $X$ to an object $(X, g)$ of $\mathcal{L}(\mathcal{V})(\Delta^{\leq \text{nop}})$, so that the map $p$ lifts to

$$p: (Y, f_Y) \to (X, g)$$

and the identity on $X$ lifts to

$$i: (X, f_X) \to (X, g).$$
This gives the diagram in $\mathbf{C}^b_{mot}(\mathcal{V})$

$$
\begin{array}{c}
\mathbb{Z}_X(q)_{m\leq*\leq n} \\
i^* \downarrow
\end{array} \xrightarrow{p^*} \mathbb{Z}_Y(q)_{m\leq*\leq n}
$$

with $i^*$ an isomorphism in $\mathbf{D}^b_{mot}(\mathcal{V})$ by (2.1.3)(e). As in (2.2.8), this defines the functor (5.2.2.3). Taking $m = 0$ gives the functor

$$
\mathbb{Z}_(-)(q): \mathcal{V}^{op}(\Delta^\leq_n) \rightarrow \mathbf{D}^b_{mot}(\mathcal{V})
$$

$$
X \mapsto \mathbb{Z}_X(q) := \mathbb{Z}_X(q)^{0\leq*\leq n}.
$$

(5.2.4)

For $n' \geq n$, and

$$X: \Delta^\leq_{n'op} \rightarrow \mathcal{V},$$

we have the canonical map

$$
\rho_{n',n}: (X, f_X) \circ i_{n',n} \rightarrow (X^\leq_n, f_X^\leq_n);
$$

this defines the natural map

$$
\rho_{n',n}^*: \mathbb{Z}_X(q)_{m\leq*\leq n'} \rightarrow \mathbb{Z}_{X^\leq_n}(q)_{m\leq*\leq n}.
$$

(5.2.5)

**5.2.3 Products**

The category $\mathcal{V}^{op}$ is a symmetric monoidal category with operation $\times_S$, and has the commutative multiplication given by the opposite of the diagonal

$$\Delta_X: X \rightarrow X \times_S X.$$  

Similarly, the category $\mathcal{L}(\mathcal{V})^{op}$ is a symmetric monoidal category with product $\times_S$, and the projection

$$\mathcal{L}(\mathcal{V})^{op} \rightarrow \mathcal{V}^{op}$$

is a symmetric monoidal functor. By the results of (5.1.1), we have the natural products

$$
\mathbb{Z}(\ast)(\cup_{(X,f),(Y,g)}): \mathbb{Z}_X(q)_{f^\leq*\leq n} \otimes \mathbb{Z}_Y(q')_{g^\leq*\leq n} \rightarrow \mathbb{Z}_{X \times_S Y}(q + q')_{f \times g^\leq*\leq n}
$$

(5.2.3.1) for $(X, f)$ and $(Y, g)$ in $\mathcal{L}(\mathcal{V})(\Delta^\leq_{n'op})$. These products are associative and graded-commutative. Taking $(X, f) = (X, f_X)$, and $(Y, g) = (Y, f_Y)$ gives the natural associative, graded-commutative products

$$
\mathbb{Z}(\ast)(\cup_{X,Y}): \mathbb{Z}_X(q)_{m\leq*\leq n} \otimes \mathbb{Z}_Y(q') \rightarrow \mathbb{Z}_{X \times_S Y}(q + q')_{m\leq*\leq n}
$$

(5.2.3.2)
Applying the functors $\mathbb{Z}(\ast)$ to the map (5.1.3.2) produces the associative, graded-commutative multiplication

$$\mathbb{Z}(\ast)(m^n(X^{m\leq s\leq n})) \colon \mathbb{Z}_X(q)^{m\leq s\leq n} \otimes \mathbb{Z}_X(q') \to \mathbb{Z}_X(q + q')^{m\leq s\leq n}.$$  \hspace{1cm} (5.2.3.3)

in $\mathcal{D}$. This makes the bi-graded $\mathbb{Z}$-module $\oplus_{p,q} H^p_{mot}(\mathbb{Z}_X(q))$ into bi-graded ring, graded-commutative in $p$; we write $H^p_{mot}(X, \mathbb{Z}(q))$ for $H^p_{mot}(\mathbb{Z}_X(q))$. For $m \leq n$, the products (5.2.3.3) make the bi-graded $\mathbb{Z}$-module $\oplus_{p,q} H^p_{mot}(\mathbb{Z}_X(q)^{m\leq s\leq n})$ a bi-graded module over $\oplus_{p,q} H^p_{mot}(X, \mathbb{Z}(q))$. Additionally, the various maps defined by changing $n$ or $m$ are ring homomorphisms, or module homomorphisms, as appropriate; this follows from the commutativity of the diagrams (5.1.3.3)-(5.1.3.5). We often write the maps (5.2.3.2) and (5.2.3.3) as $\cup_{X,Y}$ and $\cup_X$, respectively.

Let $j \colon V \to Y$ be an open simplicial subscheme of $Y$ with complement $\check{Y}$; we have the motive with supports $\mathbb{Z}_{Y,\check{Y}}(q)_g$ (2.1.2.1), given as the shifted Cone of the map $j^*$:

$$\mathbb{Z}_{Y,\check{Y}}(q)_g := \text{Cone}(j^* \colon \mathbb{Z}_Y(q)_g \to \mathbb{Z}_V(q)_j)[{-1}].$$

As the maps (5.2.3.1) are natural in $(Y, g)$, they induce the map

$$\cup_{X,Y} := \mathbb{Z}^{\leq n}(\ast)(\cup_{X,Y}) \colon \mathbb{Z}_X(q) \otimes \mathbb{Z}_{Y,\check{Y}}(q') \to \mathbb{Z}_{X \times_s Y, X \times_s \check{Y}}(q + q').$$  \hspace{1cm} (5.2.3.5)

(5.2.4) **Cycles on simplicial schemes**

Let 

$$(X, f) \colon \Delta^{\leq n}_{op} \to \mathcal{L}(\mathcal{V})$$

be a simplicial object of $\mathcal{L}(\mathcal{V})$. We have the sequence of maps in $\mathbf{C}^b_{mot}(\mathcal{V})$

$$\mathbb{Z}_X(q)^{m\leq s\leq n} \overset{\pi_m}{\longrightarrow} \mathbb{Z}_{X_m}(q) \overset{d^m}{\longrightarrow} \mathbb{Z}_{X_{m+1}}(q) \overset{f_{m+1}}{\longrightarrow},$$  \hspace{1cm} (5.2.4.1)

where $\pi_m$ is the canonical map of complexes. From (3.2.4), the sequence (5.2.4.1) gives the exact sequence

$$0 \to H^0(\mathbb{Z}_{mot}(\mathbb{Z}_X(q)^{m\leq s\leq n}[2q + m])) \longrightarrow \mathcal{Z}^q(X_m)_{f_m} \overset{\mathcal{Z}^q(d^m)}{\longrightarrow} \mathcal{Z}^q(X_{m+1})_{f_{m+1}}.$$  \hspace{1cm} (5.2.4.2)

We set

$$\mathcal{Z}^q(X)_f^{m\leq s\leq n} := H^0(\mathbb{Z}_{mot}(\mathbb{Z}_X(q)^{m\leq s\leq n}[2q + m]),$$  \hspace{1cm} (5.2.4.3)

the exact sequence (5.2.4.2) thus gives the exact sequence

$$0 \to \mathcal{Z}^q(X_f)_f^{m\leq s\leq n} \longrightarrow \mathcal{Z}^q(X_m)_f \overset{\mathcal{Z}^q(d^m)}{\longrightarrow} \mathcal{Z}^q(X_{m+1})_{f_{m+1}}.$$  \hspace{1cm} (5.2.4.4)

By (3.2.4) again, we have the canonical isomorphism

$$\text{cyc}^q_{(X,f)^{m\leq s\leq n}} \colon \mathcal{Z}^q(X_f)^{m\leq s\leq n} \to \text{Hom}_{\mathbf{K}_{mot}^b}(e \otimes 1, \mathbb{Z}_X(q)^{m\leq s\leq n}[2q + m]).$$  \hspace{1cm} (5.2.4.5)
one easily checks that, for $m = n$, this agrees with the cycle class map (3.3.2.2) (after a shift).

For $W \in \mathcal{Z}^q(X)_f^{m \leq * \leq n}$, we let

$$\text{cl}^q_{(X,f)^{m \leq * \leq n}}(W) \in \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})}(1, \mathbb{Z}_X(q)^{m \leq * \leq n}[2q + m]) = H^q_{\text{mot}}(\mathbb{Z}_X(q)^{m \leq * \leq n})$$

be the composition in $D^b_{\text{mot}}(\mathcal{V})$

$$1_{\nu_1^{-1}} \epsilon \otimes 1 \xrightarrow{\text{cyc}^q_{(X,f)^{m \leq * \leq n}}(W)} \mathbb{Z}_X(q)^{m \leq * \leq n}[2q + m],$$

where $\nu_1$ is the map (2.2.4.1). This defines the homomorphism

$$\text{cl}^q_{(X,f)^{m \leq * \leq n}}: \mathcal{Z}^q(X)_f^{m \leq * \leq n} \to H^q_{\text{mot}}(\mathbb{Z}_X(q)^{m \leq * \leq n}); \quad (5.2.4.6)$$

as above, this agrees with the shifted cycle class map (3.3.2.5) for $m = n$.

For $m \leq n' < n$, we have the canonical map of complexes

$$\pi_{m;n',n} : \mathbb{Z}_X(q)^{m \leq * \leq n} \to \mathbb{Z}_X(q)^{m \leq * \leq n'}; \quad (5.2.4.7)$$

in particular, we have the map

$$\pi_{n',n} := \pi_{0;n',n} : \mathbb{Z}_X(q) \to \mathbb{Z}_X(q)_{f \leq n'}.$$ \hspace{1cm} (5.2.4.8)

We may take $m = 0$ in (5.2.4.1)-(5.2.4.7); we write

$$\mathcal{Z}^q(X)_f := \mathcal{Z}^q(X)_f^{0 \leq * \leq n},$$

$$\text{cyc}^q_{(X,f)} := \text{cyc}^q_{(X,f)^{0 \leq * \leq n}},$$

etc. For a truncated simplicial object

$$X: \Delta^{\leq n_{\text{op}}} \to \mathcal{V}$$

we have the truncated simplicial object $(X, f_X)$ of $\mathcal{L}(\mathcal{V})$; we write

$$\mathcal{Z}^q(X)^{m \leq * \leq n} := \mathcal{Z}^q(X)^{m \leq * \leq n}_f,$$

$$\mathcal{Z}^q(X/S) := \mathcal{Z}^q(X)_{f_X},$$

$$\text{cyc}^q_X := \text{cyc}^q_{(X,f_X)},$$

etc.
For $0 \leq n' < n$, the map (5.2.4.8) induces the injective map

$$Z^q(\pi_{n',n}) : Z^q(X)_f \to Z^q(X_{\leq n'})_{f \leq n'}.$$  \hfill (5.2.4.9)

We have the map (5.1.1.1)

$$f^{m,0} : [0] \to [m].$$

(5.2.5) **Proposition**

i) The maps (5.2.4.5) and (5.2.4.6) define natural transformations of functors from $L(\Delta^{\leq n_{\text{op}}})$ to $\mathsf{Ab}$.

ii) Let

$$(X, f) : \Delta^{\leq n_{\text{op}}} \to L(\mathcal{V})$$

$$(Y, g) : \Delta^{\leq n_{\text{op}}} \to L(\mathcal{V})$$

be truncated simplicial objects of $L(\mathcal{V})$, and take

$$W_X \in Z^q(X)_f \subset Z^q(X_0)_{f_0}$$

$$W_Y \in Z^{q'}(Y)_{g^{m \leq * \leq n}} \subset Z^{q'}(Y_m)_{g_m}$$

Then the cycle $W_Y \times_{/S} X(f^{m,0})* (W_X)$ is in $Z^{q+q'}(Y)_{g^{m \leq * \leq n}}$ and we have

$$\text{cl}^q_{(X \times S, f \times g)}(W_Y \times_{/S} X(f^{m,0})* (W_X)) = \text{cl}^{q'}_{(Y, g)}(W_Y) \cup X, Y \text{ cl}^q_{(X, f)}(W_X).$$

iii) Let

$$X : \Delta^{\leq n_{\text{op}}} \to \mathcal{V}$$

be a truncated simplicial object of $\mathcal{V}$, and take

$$W \in Z^q(X) \subset Z^q(X_0)_{f_{X_0}}$$

$$W' \in Z^{q'}(X)_{m \leq * \leq n} \subset Z^{q'}(X_m)_{f_{X_m}}$$

Suppose that $X(f^{0,m})* (W)$ and $W'$ intersect properly on $X_m$. Then $W' \cdot_{X_m} X(f^{m,0})* (W)$ is defined, is in $Z^{q+q'}(X)_{m \leq * \leq n}$, and we have

$$\text{cl}^q_{X_{m \leq * \leq n}}(W' \cdot_{X_m} X(f^{m,0})* (W)) = \text{cl}^{q'}_{X_{m \leq * \leq n}}(W') \cup X, \text{ cl}^q_X(W).$$

**Proof.** The assertion (i) follows from (3.3.3) and the definitions, and (iii) follows from (ii), (i) and the definition (5.1.3.6) of $\cup_X$. The assertion (ii) follows from (1.1.6), (3.3.3), and the definition (5.1.1.3) of $\cup_{X,Y}$. \hfill $\square$
**Motives of simplicial schemes**

(5.2.6) **Cycles associated to simplicial subschemes**

Suppose we have a truncated simplicial object

\[(X, f): \Delta^{\leq n_{\text{op}}} \to \mathcal{L}(\mathcal{V})\]

of \(\mathcal{L}(\mathcal{V})\), together with a closed simplicial subscheme

\[Z \subset X\]

such that,

a) for each \(m \leq n\), \(Z_m\) is a pure codimension \(q\) subscheme of \(X_m\)

b) for each \(m \leq n\), the codimension \(q\) cycle \(|Z_m|\) determined by \(Z_m\) is in \(\mathcal{Z}^q(X_m)_{f_m}\).

c) for each map \(g: [m] \to [k]\) in \(\Delta^{\leq n}\), we have the identity of cycles

\[X(g)^*(|Z_m|) = |Z_k|\]

(note that, by (b) and (1.1.6), \(X(g)^*(|Z_m|)\) is defined).

It follows directly from (a)-(c) that the cycle \(|Z_0|\) is in \(\mathcal{Z}^q(X)_f\). We write

\[|Z| \in \mathcal{Z}^q(X)_f\]  

(5.2.6.1)

for the cycle \(|Z_0|\) considered as an element of \(\mathcal{Z}^q(X)^{\leq n}_f\). We call a subscheme \(Z\) of \(X\) a **codimension \(q\) closed subscheme of \((X, f)\)**, and the cycle (5.2.6.1) the **codimension \(q\) cycle on \((X, f)\) determined by \(Z\)**.

Let \(X\) be a simplicial scheme. By a **vector bundle of rank \(N\) on \(X\)**, we mean a map of simplicial schemes

\[p: E \to X\]

together with the structure of a vector bundle of rank \(N\) on the \(n\)-simplices

\[p_n: E_n \to X_n\]

for each \(n\), such that, for each \(g: [n] \to [m]\) in \(\Delta\), the map

\[E(g): E_m \to E_n\]

is a map of vector bundles over \((X(g), f)\), and in addition, the map

\[E_m \to X(g)^*(E_n)\]

induced by \(E(g)\) is an isomorphism. A line bundle on \(X\) is as usual a vector bundle of rank 1. Making the obvious modifications, we have the notion of a vector bundle on a truncated simplicial scheme as well.
(5.2.7) **Lemma**

Let

\[ X: \Delta^{\leq n_{op}} \to \mathcal{V} \]

be a truncated simplicial object of \( \mathcal{V} \), and let

\[ (X, f_X): \Delta^{\leq n_{op}} \to \mathcal{L}(\mathcal{V}) \]

be the lifting (5.2.2.1). Suppose we have a rank \( N \) vector bundle

\[ \pi: E \to X \]

on the truncated simplicial scheme \( X \), together with a section

\[ s: X \to E. \]

Let \( Z_m \) be the subscheme of \( X_m \) determined by \( s_m = 0 \). Suppose that

a) the subscheme \( Z_0 \) of \( X_0 \) defined by \( s_0 = 0 \) has pure codimension \( N \) on \( X_0 \).

b) the cycle \( |Z_0| \) on \( X_0 \) determined by \( Z_0 \) is in \( Z^q(X_0)_{f_{X_0}} \).

Then

i) the collection of subschemes \( Z_m \subset X_m \) determines a closed subscheme \( Z \) of \( X \)

ii) \( Z \) is a codimension \( N \) closed subscheme of \((X, f_X)\).

In particular, the cycle \( |Z_0| \) determines the element \( |Z| \) of \( Z^q(X) \).

**Proof.** Let \( g: [m] \to [k] \) be a map in \( \Delta^{\leq n} \). We have the commutative diagram

\[
\begin{array}{ccc}
E_k & \xrightarrow{E(g)} & E_m \\
\downarrow{s_k} & & \downarrow{s_m} \\
X_k & \xrightarrow{X(g)} & X_m \\
\end{array}
\]

As the map \( E(g) \) induces an isomorphism

\[ \tilde{E}(g): E_k \to X(g)^*(E_m), \]

we have the identity of subschemes of \( X_k \):

\[ Z_k = X(g)^{-1}(Z_m). \quad (1) \]

By our hypotheses (a) and (b), and the definition of \( f_X \), it follows from (1), with \( m = 0 \), that each \( Z_k \) has pure codimension \( N \) on \( X_k \). Since each \( Z_l \) is a local complete intersection in \( X_l \), and \( X_l \) is smooth over \( S \), it follows from (1) that

\[ \text{Tor}^0_{p_{X_k}}(\mathcal{O}_{Z_m}, \mathcal{O}_{X_k}) = 0 \quad (2) \]
for all $p > 0$, where $\mathcal{O}_{X_k}$ is an $\mathcal{O}_{X_m}$-module with respect the the morphism $X(g)$. Thus, from (1) and (2), we have the identity of cycles

$$|Z_k| = X(g)^*(|Z_m|).$$

(3)

Taking $m = 0$, and the assumptions (a) and (b), it follows that the cycle $|Z_k|$ is in $\mathcal{Z}^N(X_k/S)$ for all $k \leq n$. Applying this to (3) for arbitrary $m$ and $k$, and noting the definition of $(X, f_X)$, we find that $|Z_k|$ is in $\mathcal{Z}^N(X_k)_{f_{X_k}}$ for each $k$. From the definitions in (5.2.6), this completes the proof.

5.3. Chern classes of line bundles

We define the motivic first Chern class of a line bundle on a truncated simplicial scheme in $\mathcal{V}$.

(5.3.1) Line bundles

Let $p: L \to X$ be a line bundle on a scheme $X$ in $\mathcal{V}$. Combining the homotopy axiom (2.1.3)(a) and the Mayer-Vietoris sequence (2.2.6.1), we have the isomorphism in $\mathcal{D}$:

$$p^*: \mathbb{Z}_X(q) \to \mathbb{Z}_L(q).$$

By the Appendix, (A.4.9), the zero-section of $L$ gives us the cycle $0_L \in \mathcal{Z}^1(L/S)$, which in turn gives the cycle class map (5.2.4.6) in $\mathcal{D}$:

$$\text{cl}^1_X(0_L): 1 \to \mathbb{Z}_L(1)[2].$$

More generally, let

$$X: \Delta^{n\text{op}} \to \mathcal{V}$$

be a truncated simplicial object in $\mathcal{V}$, and $p: L \to X$ a line bundle on $X$. Applying (5.2.7) to the zero section of $L$, we have the element

$$0_L \in \mathcal{Z}^1(L/S).$$

We may then take the cycle class map (5.2.4.6) in $\mathcal{D}$:

$$\text{cl}^1_L(0_L): 1 \to \mathbb{Z}_L(1)[2].$$

We have as well the isomorphism in $\mathcal{D}$:

$$p^*: \mathbb{Z}_X(q) \to \mathbb{Z}_L(q).$$

(5.3.2) Definition
Let $p: L \rightarrow X$ a line bundle over $X$, with $X$ in $\mathcal{V}$. The first Chern class of $L$,

$$c_1(L) \in H^2(X, \mathbb{Z}(1)),$$

is the element corresponding to the morphism in $\mathcal{D}$

$$(p^*)^{-1} \circ \text{cl}^1(0_L): 1 \rightarrow \mathbb{Z}_X(1)[2].$$

More generally, let

$$X: \Delta^{\leq \text{nop}} \rightarrow \mathcal{V}$$

be a truncated simplicial object in $\mathcal{V}$, and $p: L \rightarrow X$ a line bundle on $X$. The first Chern class of $L$,

$$c_1(L) \in H^2(X, \mathbb{Z}(1)),$$

is the element corresponding to the morphism in $\mathcal{D}$

$$(p^*)^{-1} \circ \text{cl}^1_L(0_L): 1 \rightarrow \mathbb{Z}_X(1)[2].$$

(5.3.3) Proposition

The first Chern class satisfies

i) Functoriality: for $f: Y \rightarrow X$ a morphism in $\mathcal{V}(\Delta^{\leq \text{nop}})$, and $L$ a line bundle on $X$, we have

$$c_1(f^*(L)) = f^*(c_1(L)).$$

In addition, the simplicial first Chern class is stable in $n$, i.e., for $n' \leq n$, we have

$$\pi^*_{n,n'}(c_1(L)) = c_1(L^{\leq n'}).$$

ii) Additivity: For $L_1$ and $L_2$ line bundles on $X \in \mathcal{V}(\Delta^{\leq \text{nop}})$, we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

iii) Compatibility with divisors: Let $L$ be a line bundle on $X \in \mathcal{V}(\Delta^{\leq \text{nop}})$, and let

$$s: X \rightarrow L$$

be a section, such that the divisor $D_0$ of $s_0: X_0 \rightarrow L_0$ is in $\mathbb{Z}^1(X_0)_{fX_0}$. Let $D$ be the divisor on $(X, f_X)$ determined by the codimension one subscheme $s = 0$ of $(X, f_X)$ (see (5.2.6) and (5.2.7)). Then

$$c_1(L) = \text{cl}^1_X(D).$$
Proof. For (i), let $f_L: f^*(L) \to L$ be the canonical map of line bundles over the map $f$, giving the commutative diagram

$$
\begin{array}{ccc}
  f^*(L) & \xrightarrow{f_L} & L \\
p_Y & \downarrow & \downarrow p_X \\
Y & \xrightarrow{f} & X
\end{array}
$$

(1)

We have the identity of cycles

$$f_L^*(0_L) = 0_{f^*(L)},$$

which, from (5.2.5)(i), gives the identity

$$f_L^*(\text{cl}^1_L(0_L)) = \text{cl}^1_L f^*(L)(0_{f^*(L)}).$$

This, together with the commutativity of (1) and the definition of $c_1$, proves the first part of (i). The second part follows by a similar argument.

For (ii), we have the map over $X$

$$\pi: L_1 \times_X L_2 \to L_1 \otimes L_2$$

defined on a fiber over a point $x$ of $X^m$ by sending $(s, t)$ to the product $st$. We also have the projections

$$p_1: L_1 \times_X L_2 \to L_1; \quad p_2: L_1 \times_X L_2 \to L_2,$$

and the maps

$$p: L_1 \times_X L_2 \to X, \quad q: L_1 \otimes L_2 \to X.$$

Let $0_1$, $0_2$ and $0_{12}$ denote the zero sections on $L_1$, $L_2$ and $L_1 \otimes L_2$, respectively. Using (5.2.5)(i) repeatedly, one easily checks that

$$\begin{align*}
\pi^*(0_{12}) &= 0_1 \times_X L_2 + L_1 \times_X 0_2, \\
p_1^*(0_2) &= L_1 \times_S 0_2, \\
p_2^*(0_1) &= 0_1 \times_X L_2.
\end{align*}
$$

(3)

It follows immediately from the definition of $c_1$ and (5.2.5)(i) that

$$c_1(L_1) = (p^*)^{-1}(p_2^*(\text{cl}^1_L(0_1))); \quad c_1(L_2) = (p^*)^{-1}(p_1^*(\text{cl}^1 L_2(0_1))).$$

Since

$$c_1(L_1 \otimes L_2) = (q^*)^{-1}(\text{cl}^1_{L_1 \otimes L_2}(0_{12})), $$

and the cycle class map $\text{cl}^1$ is additive, the relations (3) prove (ii).

Finally, for (iii), it suffices to prove that

$$\text{cl}^1(p^*(D)) = \text{cl}^1_L(0_L)$$
in $H^2(L, \mathbb{Z}(1))$, where $p: L \to X$ is the structure map for the line bundle $L$.

We may form the truncated simplicial sheaf $\mathcal{O}_X(D)$ on $X$ with

$$[\mathcal{O}_X(D)]_m = \mathcal{O}_{X_m}(D_m),$$

where $D_m$ is the divisor of the section $s_m: X_m \to L_m$. We have the canonical map of truncated simplicial sheaves

$$i_D: \mathcal{O}_X \to \mathcal{O}_X(D);$$

the resulting section $s_D$ of $L$ defines by (5.2.7) and the hypothesis of (iii) a codimension one subscheme $s_D = 0$ of $(X, f_X)$ with divisor on $(X, f_X)$ equal to $D$. Pulling back by $p$, we have the section $s_1$ of $p^*(L)$ over $L$ with divisor $p^*(D)$ on $(L, f_L)$. On the other hand the identity map on $L$ determines the tautological section $s_2$ of $p^*(L)$ over $L$ with divisor $0_L$ on $(L, f_L)$.

Let $q: L \times_S \mathbb{A}^1_S \to L$ be the projection, and form the section $s_3 := tq^*(s_1) + (1-t)q^*(s_2)$ of $q^*(p^*(L))$ over $L \times_S \mathbb{A}^1_S$, where $t$ is the coordinate on $\mathbb{A}^1_S$.

Let $E$ be the divisor of the section $s_3$, and take a geometric point $a$ of $S$. For a geometric point $b \neq 0$ of $\mathbb{A}^1$ and for $m \leq n$, the restriction of $E_a$ to $L_a^m \times b$ is locally isomorphic to the graph of a function on $X_a^m$; in particular $E_a^m$ is reduced, locally irreducible and pure codimension one on $(L^m \times_S \mathbb{A}^1_S)_a$. Let

$$i_0: L \to L \times_S \mathbb{A}^1_S; \quad i_1: L \to L \times_S \mathbb{A}^1_S$$

be the 0 and 1 sections. From the Appendix, (A.4.9), and the identities

$$i_0^*(E_m) = 0_{L_m}, \quad i_1^*(E_m) = p^*(D_m)$$

of divisors on $L_m$, it follows that $E_m$ is in

$$Z^1(L_m \times_S \mathbb{A}^1_S)_{id \cup i_0 \cup i_1}$$

for each $m$. From this it follows that the subscheme of $L \times_S \mathbb{A}^1_S$ defined by the section $s_3$ is a codimension one subscheme of

$$(L \times_S \mathbb{A}^1_S, f_L \times_S \mathbb{A}^1_S \cup i_0 \cup i_1),$$

with corresponding cycle the divisor $E$.

By (4) and (5.2.5)(i), we have the identity of divisors on $(L, f_L)$

$$i_0^*(E) - i_1^*(E) = 0_L - p^*(D).$$

By homotopy axiom (2.1.3)(a), (5) implies that

$$cl^1_L(p^*(D)) = cl^1_L(0_L).$$

This gives the desired identity. \qed
(5.3.4) Remark

If \( L \to X \) is a line bundle on an \( n \)'-truncated simplicial object \( X \) of \( \mathcal{V} \), the first Chern class is stable with respect to truncation, i.e., for \( 0 \leq n < n' \), we have

\[
\rho^*_{n',n}(c_1(L)) = c_1(L^\leq_n),
\]
where \( \rho^*_{n',n} \) are the maps (5.2.2.5). This follows directly from the definition of \( c_1 \), as in the proof of (5.3.3)(i).

5.4. Projective bundle formula and Chern classes of vector bundles

We use the splitting principle to define the motivic Chern classes of vector bundles.

(5.4.1)

Let

\[
X: \Delta^{\leq n_{op}} \to \mathcal{V}
\]

be a truncated simplicial object of \( \mathcal{V} \), let \( p: E \to X \) be a rank \( N + 1 \) vector bundle on \( X \), and \( q: \mathbb{P}(E) \to X \) the associated \( \mathbb{P}^N \)-bundle. We have the tautological surjection on \( \mathbb{P}(E) \):

\[
q^*(E) \to L_E
\]

where \( L_E \) is the line bundle associate to the invertible sheaf \( \mathcal{O}(1) \) on \( \mathbb{P}(E) \).

Let \( \hat{X} \) be a simplicial closed subset of \( X \), \( \hat{P} \) the inverse image \( q^{-1}(\hat{X}) \). For each integer \( i \geq 0 \), we have the map

\[
\alpha_i^E: \mathbb{Z}_{X,\hat{X}}(q - i)[-2i] \to \mathbb{Z}_{\mathbb{P}(E),\hat{P}}(q)
\]

(5.4.1.1)
defined as the composition

\[
\mathbb{Z}_{X,\hat{X}}(q - i)[-2i] \cong \mathbb{Z}_{X,\hat{X}}(q - i)[-2i] \otimes 1 \\
\xrightarrow{id \otimes c_1(L_E)^i} \mathbb{Z}_{X,\hat{X}}(q - i)[-2i] \otimes \mathbb{Z}_{\mathbb{P}(E)}(i)[2i] \\
\xrightarrow{\cup_{\mathbb{P}(E),X}} \mathbb{Z}_{X \times_S \mathbb{P}(E),\hat{X} \times_S \mathbb{P}(E)}(q),
\]

\[
\xrightarrow{\Delta_E^* \otimes 1} \mathbb{Z}_{\mathbb{P}(E),\hat{P}}(q),
\]

where \( \Delta_E: \mathbb{P}(E) \to \mathbb{P}(E) \times_S X \) is the map \( (id, q) \), and \( \cup_{\mathbb{P}(E),X} \) is the map (5.2.3.5).

(5.4.2) Theorem

The sum

\[
\sum_{i=0}^{N} \alpha_i^E: \oplus_{i=0}^{N} \mathbb{Z}_{X,\hat{X}}(q - i)[-2i] \to \mathbb{Z}_{\mathbb{P}(E),\hat{P}}(q)
\]
is an isomorphism in \( D \).

**Proof.** By the naturality (5.3.3)(i) of \( c_1 \), the maps \( \alpha_i^E \) are natural in the triple \( (X, \hat{X}, E) \); using the definition of \( Z_{X, \hat{X}} \) and \( Z_{\mathbb{P}(E), \hat{P}} \) as shifted Cones (2.1.2.1), we reduce to the case \( \hat{X} = \emptyset \).

We now reduce to the case of an object of \( \mathcal{V} \) rather than a simplicial object, i.e., to the case \( n = 0 \). Suppose \( n > 0 \). We have the distinguished triangles in \( D \)

\[
Z_{X, n}(q)[n] \to Z_X(q) \to Z_{X, n-1}(q) \to Z_{X, n}(q)[n+1] \\
Z_{\mathbb{P}(E), n}(q)[n] \to Z_{\mathbb{P}(E)}(q) \to Z_{\mathbb{P}(E), n-1}(q) \to Z_{\mathbb{P}(E), n}(q)[n+1].
\]

From the definition of the maps (5.2.3.1), we see that the map \( E_i \) induces the map

\[
\alpha_i^E: Z_{X, n}(q - i)[n - 2i] \to Z_{\mathbb{P}(E), n}(q)[n].
\]

By (5.2.5)(ii), the definition (5.3.2) of the first Chern class, and the naturality (5.3.3)(i) of \( c_1 \), we have

\[
\alpha_i^E = \alpha_i^E[n]. \tag{1}
\]

Similarly, the map \( \alpha_i^E \) induces the map

\[
\alpha_i^{E, \leq n-1}: Z_{X, \leq n-1}(q - i)[-2i] \to Z_{\mathbb{P}(E), \leq n-1}(q);
\]

the naturality of \( c_1 \) implies

\[
\alpha_i^{E, \leq n-1} = \alpha_i^{E \leq n-1}. \tag{2}
\]

By (1) and (2), we have the map of distinguished triangles

\[
\begin{array}{ccc}
Z_{X, n}(q)[n] & \to & Z_X(q) \\
\sum_{i=0}^N \alpha_i^E & \to & \sum_{i=0}^N \alpha_i^{E \leq n-1} \\
Z_{\mathbb{P}(E), n}(q)[n] & \to & Z_{\mathbb{P}(E)}(q)
\end{array}
\]

By induction, this reduces us to the case \( n = 0, X \in \mathcal{V} \).

Using Mayer-Vietoris (2.2.6.1), and the naturality (5.3.3)(i) of \( c_1 \), we reduce to the case of trivial \( E \):

\[
E \cong \text{Spec}_{O_X}(O_X[X_0, \ldots, X_n]), \\
\mathbb{P}(E) \cong \text{Proj}_{O_X}(O_X[X_0, \ldots, X_n]).
\]

We let \( i: 0_E \to \mathbb{P}(E) \) denote the subscheme of \( \mathbb{P}(E) \) defined by \( X_1 = \ldots = X_N = 0 \), and let \( j: U \to \mathbb{P}(E) \) be the complement of \( 0_E \).

We have the projection

\[
\pi: U \to \mathbb{P}^{N-1}_X
\]
defined by
\[ \pi(x_0 : \ldots : x_N) = (x_1 : \ldots : x_N); \]
this gives \( U \) the structure of a line bundle over \( \mathbb{P}^{N-1}_X \). By the homotopy axiom, the map
\[ \pi^* : \mathbb{Z}\mathbb{P}^{N-1}_X(q) \to \mathbb{Z}U(q) \]
is an isomorphism; by induction, we have the isomorphism
\[ \sum_{i=0}^{N-1} \alpha_i^{N-1} \oplus \mathbb{Z}X(q - i)[-2i] \to \mathbb{Z}\mathbb{P}^{N-1}_X(q). \]
The naturality of \( c_1 \) implies the identity
\[ j^* \circ \alpha_i = \pi^* \circ \alpha_i^{N-1}, \]
giving us the isomorphism
\[ j^* \circ \sum_{i=0}^{N-1} \alpha_i^{N-1} \oplus \mathbb{Z}X(q - i)[-2i] \to \mathbb{Z}U(q). \] (1)

The homogeneous functions \( X_i \) define sections of \( L_E \) which are smooth over \( S \); in fact, each subscheme of \( \mathbb{P}(E) \) defined by an equation of the form
\[ X_{i_1} = \ldots = X_{i_s} = 0 \]
for \( i_1 < \ldots < i_s \) is smooth over \( S \). Thus, by the Appendix, (A.4.9), and (5.3.3)(iii), we have the identity
\[ c_1(L_E)^N = \text{cl}^N_{\mathbb{P}(E)}(0_E). \] (2)

We have the object \( \mathbb{Z}_{\mathbb{P}(E), 0_E}(q) \) (2.1.2.1) of \( C^b_{mot}(\mathcal{V}) \), defined as the shifted Cone of the morphism
\[ j^* : \mathbb{Z}_{\mathbb{P}(E)}(q) \to \mathbb{Z}U(q); \]
the Cone sequence thus gives the distinguished triangle in \( D \)
\[ \mathbb{Z}_{\mathbb{P}(E), 0_E}(q) \xrightarrow{i_{\mathbb{P}(E), 0_E}} \mathbb{Z}_{\mathbb{P}(E)}(q) \xrightarrow{j^*} \mathbb{Z}U(q). \]

We have the Gysin isomorphism (2.2.5.2)
\[ \cup[0_E] \circ q^* : \mathbb{Z}X(q - N)[-2N] \to \mathbb{Z}_{\mathbb{P}(E), 0_E}(q); \]
the identity
\[ i_{\mathbb{P}(E), 0_E} \circ (\cup[0_E] \circ q^*) = \alpha_N \]
follows from (2). This gives us the map of distinguished triangles:

\[
\begin{array}{c}
\mathbb{Z}_{\mathbb{P}(E), 0} (q) \\
\cong \mathbb{Z}_{\mathbb{P}(E)} (q) \\
\mathbb{Z}_{U} (q)
\end{array}
\]

\[
\begin{array}{c}
\cup [0, q] \\
\sum_{i=0}^{N} \alpha_i \\
\sum_{i=0}^{N-1} \alpha_i
\end{array}
\]

\[
\begin{array}{c}
\mathbb{Z}_X (q - N)[-2N] \\
\oplus \mathbb{Z}_X (q - i)[-2i] \\
\oplus \mathbb{Z}_X (q - i)[-2i]
\end{array}
\]

As the two maps on the ends are isomorphisms, the map in the middle is an isomorphism as well, completing the proof.

(5.4.3) Definition

Let

\[X: \Delta \leq \text{op} \to \mathcal{V}\]

be a truncated simplicial object of \(\mathcal{V}\), let \(\tilde{X}\) be a closed simplicial subscheme of \(X\), and let

\[E \to X\]

be a vector bundle of rank \(N + 1\) on \(X\). Let \(q: \mathbb{P}(E) \to X\) be the associated projective bundle with tautological quotient line bundle \(L_E\), and let \(\zeta = c_1 (L_E)\). The Chern classes of \(E\) are the elements \(c_i (E) \in H^{2i} (X, \mathbb{Z}(i))\) satisfying

\[
\sum_{i=0}^{N} (-1)^i q^* (c_i (E)) \zeta^{N-i} = 0, \ c_0 (E) = 1.
\]

(1)

By (5.4.2), the \(c_i (E)\) exist and are uniquely determined by the identity (1). We define the total Chern class \(c(E)\) to be the sum

\[c(E) = \sum_{i=0}^{N} c_i (E).\]

(5.4.4) Theorem

The Chern classes satisfy

i) Naturality: let \(f: Y \to X\) be a morphism in \(\mathcal{V}(\Delta \leq \text{op})\), \(E\) a vector bundle on \(X\). Then

\[f^* (c(E)) = c(f^* (E)).\]

Similarly if we have \(X\) in \(\mathcal{V}(\Delta \leq n' \text{op})\), \(E\) a vector bundle on \(X\) and \(0 \leq n < n'\), then, via the maps (5.2.2.5)

\[\rho_{n', n}: \mathbb{Z}_X (q) \to \mathbb{Z}_{X \leq n}(q),\]
we have

$$
\rho_{n^*,n}^*(c(E)) = c(E^{\leq n}).
$$

ii) Normalization: The two definitions ((5.3.2) and (5.4.3)) of the first Chern class of a line bundle agree.

Proof. The first part of (i) follows from the functoriality of the first Chern class (5.3.3)(i), and the naturality of the projective bundle isomorphism of (5.4.2); the second part follows similarly using (5.3.4), and noting that the projective bundle isomorphism (5.4.2) is compatible with truncation. The statement (ii) follows from the defining relation for $c_1$ of a line bundle $L \to X$ in (5.4.3):

$$
c_1(L) - \zeta = 0
$$

and the identification of the tautological line bundle $L_L$ on $\mathbb{P}(L) = X$ with $L$.

(5.4.5) Remark

If $E \to X$ is a rank $r$-vector bundle on a scheme $X \in \textbf{Sm}_S$, we may take a trivializing open cover $U = \{U_0, \ldots, U_N\}$ for $E$, and form the Čech simplicial scheme over $X$:

$$
U_s \to X
$$

A choice of transition matrices $\{g_{ij}\}$ for $E$ with respect to the covering $U$ gives the map of simplicial schemes over $S$:

$$
g: U_s \to \text{BGL}_r/S.
$$

If we let $E_r$ be the canonical rank $r$ vector bundle on $\text{BGL}_r/S$, we then have the truncated Chern classes

$$
c_i(E_r^{\leq n}) \in H^{2i}(\mathbb{Z}_{\text{BGL}_r/S}(i)^{\leq n}).
$$

We may then pull back the $c_i$ via $g$ to give classes

$$
g^*(c_i(E_r^{\leq n})) \in H^{2i}(\mathbb{Z}_{U_s(i)^{\leq n}}).
$$

On the other hand, the augmentation induces an isomorphism in $\mathcal{DM}(\mathcal{V})$

$$
\mathbb{Z}_X(i) \to \mathbb{Z}_{U_s(i)^{\leq n}}
$$

for all $n \geq N$. Thus, we get the elements

$$
g^*(c_i(E_r^{\leq N})) \in H^{2i}(\mathbb{Z}_X(i)) = H^{2i}(X, \mathbb{Z}(i)).
$$

It follows from the naturality of the Chern classes that

$$
g^*(c_i(E_r^{\leq N})) = c_i(E);
$$

$i = 0, 1, \ldots$
(5.4.6) Theorem (Whitney Product Formula)

Let $X$ be an $n$-truncated simplicial object in $\mathcal{V}$, and

$$0 \to E_1 \to E \to E_2 \to 0$$

an exact sequence of vector bundles on $X$. Then

$$c(E) = c(E_1)c(E_2).$$

Proof. Let $q: \mathcal{F} \to X$ be a flag bundle over $X$, associated to a vector bundle $F$ on $X$. As $\mathcal{F}$ can be constructed as a sequence of projective bundles over $X$, the pull-back

$$q^*: H^p(X, \mathbb{Z}(q)) \to H^p(\mathcal{F}, \mathbb{Z}(q))$$

is injective, i.e. the splitting principle holds for motivic cohomology. Replacing $X$ with the appropriate flag bundle over $X$, and changing notation, we change to the following problem:

Suppose $E \to X$ is a vector bundle on $X$, with a filtration

$$\{0\} = E_0 \subset E_1 \subset \ldots \subset E_N = E$$

such that the quotients $E_k/E_{k-1}$ are line bundles $L_k$ on $X$. Then

$$c(E) = \prod_{k=1}^N (1 + c_1(L_k)).$$

Pulling back further to the (affine) bundle of splittings of the above filtration, we reduce to showing:

$$c(\oplus_{k=1}^N L_k) = \prod_{k=1}^N (1 + c_1(L_k)).$$

Let $E = \oplus_{k=1}^N L_k$, and let $q: \mathbb{P} \to X$ be the projective bundle $\mathbb{P}(E)$. By the discussion in (5.2.6), each sub-bundle $L_k$ defines a divisor $D_k$ of $\mathbb{P}$, isomorphic to the projective bundle $\mathbb{P}(\oplus_{j \neq k} L_j) \subset \mathbb{P}$. One easily sees that line bundle $N_k$ on $\mathbb{P}$ associated to $D_k$ is isomorphic to $q^*(L_k^{-1}) \otimes \mathcal{O}(1)$. Thus, by (5.3.3)(iii), we have

$$\text{cl}^1(D_k) = \zeta - q^*(c_1(L_k)),$$

where $\zeta = c_1(\mathcal{O}(1))$. Since the intersection $D_1 \cap \ldots \cap D_N$ is empty on $\mathbb{P}$, we have by
(5.2.5)(iii)
\[
0 = cl^N(D_1 \cap \ldots \cap D_N) \\
= cl^1(D_1) \cup \ldots \cup cl^1(D_N) \\
= \prod_{k=1}^{N}(\zeta - q^*(c_1(L_k))) \\
= \zeta^N + \sum_{k=1}^{N}(-1)^k\zeta^{N-k}q^*(\sigma_k),
\]
where $\sigma_k$ is the $k$th symmetric function in the Chern classes $c_1(L_1), \ldots, c_1(L_N)$. By the defining relation (5.4.3) for the Chern classes of $E$, this shows
\[
c_k(E) = \sigma_k,
\]
i.e.
\[
c(\oplus_{k=1}^{N}L_k) = \prod_{k=1}^{N}(1 + c_1(L_k)),
\]
as desired.

\section{5.5. Chern classes for higher $K$-theory}

We use the method of Gillet [G] to define motivic Chern classes for higher $K$-theory.

\subsection*{(5.5.1) Open covers}

Let $X$ be in $\mathcal{V}$, and $\mathcal{U} = \{U_0, \ldots, U_m\}$ an open cover. We have the associated augmented simplicial scheme
\[
\epsilon: N(\mathcal{U})_* \rightarrow X
\]
with $n$-simplices the disjoint union of the intersections
\[
U_{i_0} \cap \ldots \cap U_{i_n}.
\]
As in (4.2.2), we let $\mathcal{Z}_{\mathcal{U}}(q)^{* \leq n}$ be the object of $\mathbf{C}^b_{mot}(\mathcal{V})$ defined as the sub-complex of $\mathcal{Z}_{U_*}(q)^{* \leq n}$ generated by the non-degenerate simplices; i.e., those intersections (5.5.1.1) with distinct indices $i_0, \ldots, i_n$. We note that $\mathcal{Z}_{\mathcal{U}}(q)^{* \leq n} = \mathcal{Z}_{\mathcal{U}}(q)^{* \leq n'}$ for all $n' \geq n \geq m+1$; we denote the complex $\mathcal{Z}_{\mathcal{U}}(q)^{* \leq n}$ for $n \geq m + 1$ by $\mathcal{Z}_{\mathcal{U}}(q)$. The Mayer-Vietoris sequence (2.2.6.1) shows that the augmentation induces an isomorphism
\[
\epsilon^*: \mathcal{Z}_X(q) \rightarrow \mathcal{Z}_{\mathcal{U}}(q).
\]

\subsection*{(5.5.2) Homology of $GL_N$}
Now suppose each $U_k$ is affine:

$$U_k = \text{Spec}(A_k).$$

We may form the standard chain complex $C_*(\GL_N(A_k), \mathbb{Z})$ which computes the group homology:

$$H_p(C_*(\GL_N(A_k), \mathbb{Z})) = H_p(\GL_N(A_k), \mathbb{Z}).$$

We may also form the simplicial set $\text{Hom}_S(U_k, \text{BGL}_N/S)$ and the associated chain complex

$$Z[\text{Hom}_S(U_k, \text{BGL}_N/S)]_*;$$

we have the canonical isomorphism

$$C_*(\GL_N(A_k), \mathbb{Z}) \cong Z[\text{Hom}_S(U_k, \text{BGL}_N/S)]_* \quad (5.5.2.1)$$

More generally, we may apply the functor $A \mapsto C_*(\GL_N(A), \mathbb{Z})$ to the non-degenerate portion of the cosimplicial ring $A_*$ associated to the simplicial affine scheme $U_*$, and take the corresponding total complex, forming the Čech complex $C^{n,d}_*(\GL_N(A_*), \mathbb{Z})$. We define $Z[\text{Hom}_S(N(U)_*^{n,d}, \text{BGL}_N/S)]_*$ similarly as the complex gotten by applying the functor $U \mapsto Z[\text{Hom}_S(U, \text{BGL}_N/S)]_*$ to the non-degenerate portion of the simplicial scheme $N(U)_*$ and taking the total complex. The isomorphism (5.5.2.1) extends to the isomorphism of total complexes

$$C^{n,d}_*(\GL_N(A_*), \mathbb{Z}) \to Z[\text{Hom}_S(N(U)_*^{n,d}, \text{BGL}_N/S)]_* \quad (5.5.2.2)$$

**(5.5.3) Homology and Motivic Cohomology**

For all $n \geq m + 1$, we have the canonical map

$$i^*_p: H_p(Z[\text{Hom}_S(N(U)_*^{n,d}, \text{BGL}_N/S)]_*) \to \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})}(Z_{\text{BGL}_N/S}(q)^{\leq n}, Z_{U}(q)[-p])$$

induced by the identity for all $\Gamma(S, \mathcal{O}_S)$-algebras $A$:

$$\GL_{N+1}^p(A) = \text{Hom}_S(\text{Spec}(A), (\GL_N/S)^{p+1}).$$

In addition, the maps $i^*_p$ are compatible with taking refinements of the affine cover: if $i: \mathcal{V} \to \mathcal{U}$ is an affine refinement of the affine cover $\mathcal{U}$, the diagram

$$\begin{array}{ccc}
H_p(Z[\text{Hom}_S(N(U)_*^{n,d}, \text{BGL}_N/S)]_*) & \xrightarrow{i^*_p} & \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})}(Z_{\text{BGL}_N/S}(q)^{\leq n}, Z_{U}(q)[-p]) \\
\downarrow i^* & & \downarrow i^* \\
H_p(Z[\text{Hom}_S(N(\mathcal{V})_*^{n,d}, \text{BGL}_N/S)]_*) & \xrightarrow{i^*_p} & \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})}(Z_{\text{BGL}_N/S}(q)^{\leq n}, Z_{\mathcal{V}}(q)[-p])
\end{array}$$
commutes for all \( n \) sufficiently large. Thus, if we define \( H_p(X, \mathcal{G}L_N/S; \mathbb{Z}) \) by

\[
H_p(X, \mathcal{G}L_N/S; \mathbb{Z}) = \lim_{\rightarrow} H_p(\mathbb{Z}[\text{Hom}_S(N(U)^{n.d.}, \mathcal{G}L_N/S)])_* \)

we have the map

\[
H_p^{X,N}(X, \mathcal{G}L_N/S; \mathbb{Z}) \rightarrow \lim_n \text{Hom}_{\mathcal{D}_{\text{mot}}(\mathcal{V})}(\mathbb{Z}_{\mathcal{G}L_N/S}(q)^{\leq n}, \mathbb{Z}_X(q)[-p])
\]

induced by the composition \((\epsilon^*)^{-1} \circ i^{U_*}_p\). The maps \( H^{X,N}_p \) are stable in \( N \) in the obvious sense.

(5.5.4) Universal Chern classes

Let \( p_n: E_N \rightarrow \mathcal{G}L_N/S \) be the tautological rank \( N \) vector bundle over the simplicial \( S \)-scheme \( \mathcal{G}L_N/S \). We have the inclusion

\[
i_N: \mathcal{G}L_N/S \rightarrow \mathcal{G}L_{N+1}/S
\]

defined by

\[
i_N(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.
\]

In addition, we have \( i^*_N(E_{N+1}) \cong E_N \oplus 1 \), where 1 denotes the trivial line bundle. Thus, by the Whitney product formula (5.4.6) and the stability of Chern classes (5.4.4)(i), we have

\[
i^*_{n',n,N}(c(E_{N+1}^{\leq n'})) = c(E_N^{\leq n})
\]

for all \( n' \geq n \geq 0 \).

For each \( N, n \) and \( q \) we may consider \( c_q(E_N)^{\leq n} \) as a map

\[
c_q(E_N)^{\leq n}: 1 \rightarrow \mathbb{Z}_{\mathcal{G}L_N/S}(q)^{\leq n}[2q]
\]

in \( \mathcal{D}_{\text{mot}}(\mathcal{V}) \). Thus, if \( z \) is in \( H_p(X, \mathcal{G}L_N/S; \mathbb{Z}) \), we may form the composition

\[
H_p^{X,n,N}(z)[2q] \circ c_q(E_N)^{\leq n}: 1 \rightarrow \mathbb{Z}_X(q)[2q - p],
\]

for all \( n \) sufficiently large, defining the element \( c_{n,N}^{q,2q-p}(z) \) of \( H^{2q-p}(X, \mathbb{Z}(q)) \).

(5.5.5) Lemma

i) If \( z \) is in the image of \( H_p(\mathbb{Z}[\text{Hom}_S(N(U)^{n.d.}, \mathcal{G}L_M/S)])_* \) for some affine open cover

\[
U = \{U_0, \ldots, U_m\}
\]

of \( X \), then \( c_{n,N}^{q,2q-p}(z) \) is independent of \( n \) for all \( n \geq m + p + 1 \).
ii) If \( z = i_{N-1}(z') \), for some \( z' \in H_p(X, \mathcal{B}\mathcal{L}_{N-1}/S; \mathbb{Z}) \), then
\[
c_{n,N}^{q, 2q-p}(z) = c_{n,N-1}^{q, 2q-p}(z')
\]

**Proof.** The first assertion follows from the fact that
\[
Z_{N(U_n)}(q)^* \leq n = Z_{N(U_n)}(q)^* \leq n'
\]
for all \( n' \geq n \geq m + 1 \), the naturality (5.4.4)(i), and the stability result (5.5.4.1). The second follows from (5.5.4.1). \( \square \)

**5.5.6 Chern classes for higher K-theory**

If we define \( H_p(X, \mathcal{G}\mathcal{L}/S; \mathbb{Z}) \) as the direct limit
\[
H_p(X, \mathcal{G}\mathcal{L}/S; \mathbb{Z}) = \lim_{\rightarrow N} H_p(X, \mathcal{G}\mathcal{L}_N/S; \mathbb{Z}),
\]

it follows from (5.5.5) that we have a well-defined map
\[
H^{c_{n,N}^{q, 2q-p}}: H_p(X, \mathcal{G}\mathcal{L}/S; \mathbb{Z}) \to H^{2q-p}(X, \mathbb{Z}(q)). \tag{5.5.6.1}
\]

Following the arguments of Gillet in [G], Suslin’s stability result [Su] shows that the Hurewicz morphisms
\[
h_p^N(A): K_p(A) = \pi_p(\mathcal{B}\mathcal{L}_N^+(A)) \to H_p(\mathcal{B}\mathcal{L}_N^+(A); \mathbb{Z}) = H_p(\mathcal{G}\mathcal{L}_N(A); \mathbb{Z})
\]
give rise to a well-defined map
\[
h_p^X: K_p(X) \to H_p(X, \mathcal{G}\mathcal{L}/S; \mathbb{Z}). \tag{5.5.6.2}
\]

Composing (5.5.6.1) with (5.5.6.2) gives the Chern class map
\[
c_{n,N}^{q, 2q-p}: K_p(X) \to H^{2q-p}(X, \mathbb{Z}(q)). \tag{5.5.6.3}
\]

It follows from the (5.4.5) that \( c_{n,N}^{q, 2q} \) is an extension to \( K_0(X) \) of the Chern class of vector bundles \( c_q \).
Chapter 6
Push-forward

In this chapter, we define the push-forward morphism in $\mathcal{DM}(S)$ associated to a projective
morphism in $\text{Sm}_S$, and verify the properties normally satisfied by projective push-forward
in a reasonable cohomology theory: functoriality, compatibility with pull-back in cartesian
squares, and the projection formula. We also verify the compatibility with cycle classes
for the case of a closed embedding; the compatibility of projective push-forward with cycle
classes for an arbitrary projective morphism is also valid for $S$ of the form $\text{Spec}(k)$, where
$k$ is a field of characteristic zero, and is valid with rational coefficients in case $\text{char}(k) > 0$.

6.1. The Gysin morphism

We use Fulton’s method of “deformation to the normal bundle” to define the Gysin mor-
phism associated to a closed embedding.

(6.1.1) The deformation diagram and the Gysin morphism

Let $i: Z \rightarrow X$ be a closed codimension $d$ embedding in $\text{Sm}_S$, and let $\hat{Z}$ a be a closed subset
of $Z$. Let

$$q: Y \rightarrow X \times_S \mathbb{A}_S^1$$

be the blow-up of $X \times_S \mathbb{A}_S^1$ along $Z \times 1$, $\hat{Y}$ the proper transform of $\hat{Z} \times_S \mathbb{A}_S^1$ to $Y$, $P$ the
full inverse image of $Z \times 1$ in $Y$, $\hat{P}$ the closed subset $P \cap \hat{Y}$ of $P$. Let $i_0: X \rightarrow Y$ be the
composition of the inclusion

$$\text{id}_X \times j_0: X \rightarrow X \times_S \mathbb{A}_S^1,$$

$$x \mapsto (x, 0),$$

with the inverse of the blow-up $q: Y \rightarrow X \times_S \mathbb{A}_S^1$, $i_1: P \rightarrow Y$ the inclusion. $P$ is isomorphic
to the projectivization of the normal bundle of $Z \times 1$ in $X \times_S \mathbb{A}_S^1$, let $f: P \rightarrow Z$ be the
resulting projection. Let $[Z \times \mathbb{A}^1]$ denote the proper transform of $Z \times \mathbb{A}^1$ to $Y$. We note
that the restriction of $f$ to $[Z \times \mathbb{A}^1]$ gives an isomorphism

$$[Z \times \mathbb{A}^1] \rightarrow Z \times \mathbb{A}^1,$$

determining sections $s': Z \times \mathbb{A}^1$ to $q$ over $Z \times \mathbb{A}^1$, and $s: Z \rightarrow P$ to $f$ over $Z$, with $s(\hat{Z}) = \hat{P}$.

We encapsulate the above discussion in the following diagram:

$$
\begin{array}{cccccccc}
X & \xrightarrow{i_0} & Y & \leftarrow & \xrightarrow{i_1} & P \\
\| & & q \downarrow & \leftarrow & s' & \uparrow s \uparrow f \\
X \times 0 & \xrightarrow{id_X \times j_0} & X \times \mathbb{A}^1 & \leftarrow & Z \times \mathbb{A}^1 & \leftarrow & Z = Z \times 1
\end{array}
$$

(6.1.1.1)
where \(j_0:0 \to A^1, j_1:1 \to A^1\) are the inclusions.

The map (2.2.5.2)

\[
\cup [s(Z)] \circ f^*: \mathbb{Z}_{\hat{Z}}(-d)[-2d] \to \mathbb{Z}_{\hat{P},\hat{\cdot}}
\]

is an isomorphism by (2.1.3)(d) and (2.2.5); the maps

\[
i_1^*: \mathbb{Z}_{Y,\hat{Y}} \to \mathbb{Z}_{P,\hat{\cdot}}; \quad i_0^*: \mathbb{Z}_{Y,\hat{Y}} \to \mathbb{Z}_{X,\hat{Z}}
\]

are isomorphisms by the homotopy axiom (2.1.3)(a). This gives the sequence of isomorphisms in \(\mathcal{D}\mathcal{M}(S)\):

\[
\mathbb{Z}_{\hat{Z}}(-d)[-2d] \xrightarrow{\cup [s(Z)] \circ f^*} \mathbb{Z}_{P,\hat{\cdot}} \xrightarrow{(i_1^*)^{-1}} \mathbb{Z}_{Y,\hat{Y}} \xrightarrow{i_0^*} \mathbb{Z}_{X,\hat{Z}};
\]

we denote the composition by

\[
i_*: \mathbb{Z}_{Z,\hat{Z}}(-d)[-2d] \to \mathbb{Z}_{X,\hat{Z}}.
\]

For \(\hat{Z} = Z\), this gives the isomorphism

\[
i_*: \mathbb{Z}_Z(-d)[-2d] \to \mathbb{Z}_{X,Z}.
\]

More generally, if \(\hat{X}\) is a closed subset of \(X\) containing \(\hat{Z}\), we denote the composition

\[
\mathbb{Z}_{Z,\hat{Z}}(-d)[-2d] \xrightarrow{\cup [s(Z)] \circ f^*} \mathbb{Z}_{P,\hat{\cdot}} \xrightarrow{(i_1^*)^{-1}} \mathbb{Z}_{Y,\hat{Y}} \xrightarrow{i_0^*} \mathbb{Z}_{X,\hat{\cdot}}
\]

by

\[
i_*: \mathbb{Z}_{Z,\hat{Z}}(-d)[-2d] \to \mathbb{Z}_{X,\hat{\cdot}}.
\] (6.1.1.2)

(6.1.2) The Gysin distinguished triangle

If \(j: U \to X\) is an open subscheme with \(Z \subset U\), it follows from the excision axiom (2.1.3)(b) that the diagram of isomorphisms

\[
\begin{array}{ccc}
\mathbb{Z}_{Z,\hat{Z}}(-d)[-2d] & \xrightarrow{i_*} & \mathbb{Z}_{X,\hat{Z}} \\
\| & & \downarrow j^* \\
\mathbb{Z}_{Z,\hat{Z}}(-d)[-2d] & \xrightarrow{i_*} & \mathbb{Z}_{U,\hat{Z}}
\end{array}
\]

commutes. Thus, if \(p: X' \to X\) is a morphism which is étale over a neighborhood of \(Z\), and admits a section over \(Z\), \(s: Z \to X'\), we have a commutative diagram of isomorphisms

\[
\begin{array}{ccc}
\mathbb{Z}_{Z,\hat{Z}}(-d)[-2d] & \xrightarrow{i_*} & \mathbb{Z}_{X,\hat{Z}} \\
\mathbb{Z}_{s(Z),s(\hat{Z})}(-d)[-2d] & \xrightarrow{i'_*} & \mathbb{Z}_{X',s(\hat{Z}})
\end{array}
\] (6.1.2.1)
Combining the isomorphism (6.1.1.2) with the distinguished triangle

\[ \mathbb{Z}_{X,\hat{Z}} \to \mathbb{Z}_X \to \mathbb{Z}_{X-\hat{Z}} \to \mathbb{Z}_{X,\hat{Z}}[1] \]

gives the Gysin distinguished triangle

\[ \mathbb{Z}_{\hat{Z}}(-d)[-2d] \to \mathbb{Z}_X \to \mathbb{Z}_{X-\hat{Z}} \to \mathbb{Z}_{\hat{Z}}(-d)[1-2d]; \quad (6.1.2.2) \]

the long exact local cohomology sequence associated to (6.1.2.2) (after twisting by \( \mathbb{Z}(q) \))
gives the Gysin sequence

\[ \ldots \to H^{p-2d}_{\hat{Z}}(Z, \mathbb{Z}(q-d)) \to H^p(X, \mathbb{Z}(q)) \to H^p(X - \hat{Z}, \mathbb{Z}(q)) \]
\[ \to H^{p-2d-1}_{\hat{Z}}(Z, \mathbb{Z}(q-d)) \to \ldots. \quad (6.1.2.3) \]

In particular, for \( \hat{Z} = Z \), we have the Gysin distinguished triangle

\[ \mathbb{Z}_Z(-d)[-2d] \to \mathbb{Z}_X \to \mathbb{Z}_{X-Z} \to \mathbb{Z}_Z(-d)[1-2d] \]

and the Gysin exact sequence

\[ \ldots \to H^{p-2d}(Z, \mathbb{Z}(q-d)) \to H^p(X, \mathbb{Z}(q)) \to H^p(X - \hat{Z}, \mathbb{Z}(q)) \]
\[ \to H^{p-2d-1}(Z, \mathbb{Z}(q-d)) \to \ldots. \]
6.2. Properties of the Gysin morphism

(6.2.1) Proposition

Suppose we have subschemes

\[ W \xrightarrow{i} Z \xrightarrow{j} X \]

of a scheme \( X \), with \( X \), \( Z \) and \( W \) in \( \text{Sm}_S \), and with

\[ \text{codim}_X(Z) = d; \quad \text{codim}_Z(W) = e. \]

Let \( \hat{W} \) be a closed subset of \( W \), \( \hat{Z} \) a closed subset of \( Z \), with \( \hat{Z} \subset \hat{W} \). Then the diagrams

\[
\begin{align*}
Z_{W,\hat{W}}(-d-e)[-2d-2e] & \xrightarrow{(i\circ j)_*} Z_{Z,\hat{Z}}(-d)[-2d] \\
& \xrightarrow{j_*} Z_{X,\hat{W}} \\
& \xrightarrow{i_*} Z_{X,\hat{W}} \\
\end{align*}
\]

and

\[
\begin{align*}
Z_{Z,\hat{Z}}(-d)[-2d] & \xrightarrow{i_*} Z_{X,\hat{W}} \\
& \xrightarrow{id_X} Z_{X,\hat{W}} \\
\end{align*}
\]

commute in \( \mathcal{DM}(S) \). In addition, if \( i = \text{id}_X \), then \( i_* = \text{id} \).

Proof: The commutativity of (ii) follows directly from the functoriality of the cycle map, (3.3.3)(i).

To prove the assertion \( \text{id}_{X*} = \text{id} \), we note that the blow-up of \( X \times \mathbb{A}^1 \) along \( X \times 1 \) is isomorphic to \( X \times \mathbb{A}^1 \), hence \( \text{id}_{X*} \) is the composition

\[
Z_{X,\hat{X}} \xrightarrow{(i_0)^{-1}} Z_{X \times \mathbb{A}^1, \hat{X} \times \mathbb{A}^1} \xrightarrow{i_0^*} Z_{X,\hat{X}},
\]

where \( i_0 : X \to X \times \mathbb{A}^1 \) and \( i_1 : X \to X \times \mathbb{A}^1 \) are the 0 and 1 sections. As \( p_1 \circ i_0 = p_1 \circ i_1 = \text{id}_X \), the above composition is the identity, completing the proof.

For (i), let

\[
\begin{align*}
Y_W, \hat{Y}_W, P_W, \hat{P}_W, i_{W0} : X & \to Y_W, i_{W1} : P_W \to Y_W, \\
f_W : P_W & \to W, [W \times A^1]_W \text{ and } s_W : W \to P_W
\end{align*}
\]

be as in (6.1.1) with \( \hat{W} \) replacing \( \hat{Z} \), and \( W \) replacing \( Z \). Similarly, let

\[
\begin{align*}
Y^Z_W, \hat{Y}^Z_W, P^Z_W, \hat{P}^Z_W, i^Z_{W0} : X & \to Y^Z_W, i^Z_{W1} : P^Z_W \to Y^Z_W, \\
f^Z_W : P^Z_W & \to W, [W \times A^1]_W \text{ and } s^Z_W : W \to P^Z_W
\end{align*}
\]
be as in (6.1.1), after replacing $\hat{Z}$ with $\hat{W}$, replacing $X$ with $Z$ and replacing $Z$ with $W$.

Finally, let

$$Y_Z, \hat{Y}_Z, P_Z, \hat{P}_Z, i_{Z0}: X \to Y_Z, i_{Z1}: P \to Y_Z,$$

$$f_Z: P_Z \to Z, [Z \times A^1]_Z \text{ and } s_Z: Z \to P_Z$$

be as in (6.1.1), with $\hat{W}$ replacing $\hat{Z}$ (and leaving $X$ and $Z$ the same).

We have the regular subscheme $T := s_Z(W)$ of $P_Z$, and closed subset $\hat{T} := s_Z(\hat{W})$ of $T$. Let $i_T: T \to P_Z$ be the inclusion. Let

$$Y_T, \hat{Y}_T, P_T, \hat{P}_T, i_{T0}: P_Z \to Y_T, i_{T1}: P \to Y_T,$$

$$f_T: P_T \to T, [T \times A^1]_T \text{ and } s_T: T \to P_T$$

be as in (6.1.1), after replacing $X$ with $P_Z$, replacing $Z$ with $T$ and replacing $\hat{Z}$ with $\hat{T}$.

The section $s_Z: Z \to P_Z$ to $f_Z$ gives the section

$$s_Z \times id: Z \times A^1 \to P_Z \times A^1$$

to $f_Z \times id_{A^1}$; blowing up $W \times 1$ and $s(W) \times 1$ gives the section

$$s_{Z/W}: Y^Z_W \to Y_T$$

to the map

$$f_{Z/W}: Y_T \to Y^Z_W$$

induced by $f_Z \times id_{A^1}$.

Restricting $s_{Z/W}$ to $P^Z_W$ gives the commutative diagram

$$\begin{array}{ccc}
P_Z^W & \xrightarrow{s_{Z/W}} & P_T \\
\downarrow f_Z^W & \quad & \downarrow s_T \\
W & \xrightarrow{s_W} & T \\
\end{array}$$

This gives us the commutative diagram of isomorphisms

$$\begin{array}{cc}
Z_{P_Z,T} & \xrightarrow{(i_{Z0})^{-1}} \\
\downarrow s_Z^* \\
Z_{Z/W, T} & \xrightarrow{(i_{W1})^{-1}} \\
\downarrow s_{Z/W}^* \\
Z_{Z/W} & \xrightarrow{(i_{W0})^{-1}} \\
\downarrow s_{Z/W}^* \\
Z_{P_W, P_T} & \xrightarrow{\cup [s_T(T)] \circ f_T^*} \\
\downarrow s_T^* \\
Z_{T, T}(-e)[-2e] \\
\end{array}$$

(1)

Let

$$h_1: Q_1 \to X \times A^1 \times A^1$$

be the blow-up of $X \times A^1 \times A^1$ along the subscheme $Z \times A^1 \times 1$, let

$$[W \times 1 \times A^1]_1 \subset Q_1$$
be the proper transform of \( W \times 1 \times \mathbb{A}^1 \), and let

\[
h_2: Q \to Q_1
\]

be the blow-up along \([W \times 1 \times \mathbb{A}^1]_1\). Let

\[
h: Q \to X \times \mathbb{A}^1 \times \mathbb{A}^1
\]

the composition \( h_1 \circ h_2 \). Then we have isomorphisms (as \( X \times \mathbb{A}^1\)-schemes)

\[
\begin{align*}
  h^{-1}(X \times \mathbb{A}^1 \times 0) & \cong Y_W; \\
  h^{-1}(X \times 0 \times \mathbb{A}^1) & \cong Y_Z. 
\end{align*}
\]

(2)

We identify \( h^{-1}(X \times \mathbb{A}^1 \times 0) \) with \( Y_W \), and \( h^{-1}(X \times 0 \times \mathbb{A}^1) \) with \( Y_Z \) via these isomorphisms.

Let \( E_1 \) be the exceptional divisor of \( h_1 \), \( E \) the exceptional divisor of \( h_2 \), and \([E_1]\) the proper transform of \( E_1 \) to \( Q \). Then we have an isomorphism

\[
[E_1] \cong Y_T
\]

(3)

as a \( Z \times \mathbb{A}^1\)-scheme; we identify \([E_1]\) with \( Y_T \) via this isomorphism.

Let

\[
[W \times \mathbb{A}^1 \times \mathbb{A}^1], \quad [\hat{W} \times \mathbb{A}^1 \times \mathbb{A}^1]
\]

be the proper transform of \( W \times \mathbb{A}^1 \times \mathbb{A}^1 \), \( \hat{W} \times \mathbb{A}^1 \times \mathbb{A}^1 \) to \( Q \). Then we have

\[
\begin{align*}
  h^{-1}(X \times \mathbb{A}^1 \times 0) \cap [W \times \mathbb{A}^1 \times \mathbb{A}^1] & = [W \times \mathbb{A}^1]_W; \\
  h^{-1}(X \times 0 \times \mathbb{A}^1) \cap [W \times \mathbb{A}^1 \times \mathbb{A}^1] & = [W \times \mathbb{A}^1]_Z; \\
  \text{and} \\
  [E_1] \cap [W \times \mathbb{A}^1 \times \mathbb{A}^1] & = [W \times \mathbb{A}^1]_T.
\end{align*}
\]

(4)

We have similar equalities replacing \( W \) with \( \hat{W} \). We further note that the map

\[
h_{|[W \times \mathbb{A}^1 \times \mathbb{A}^1]}: [W \times \mathbb{A}^1 \times \mathbb{A}^1] \to W \times \mathbb{A}^1 \times \mathbb{A}^1
\]

is an isomorphism.

Let \( \hat{Q} = [\hat{W} \times \mathbb{A}^1 \times \mathbb{A}^1] \) and \( \hat{E} = E \cap [\hat{W} \times \mathbb{A}^1 \times \mathbb{A}^1] \). Let

\[
h_E: E \to W \times \mathbb{A}^1
\]

be the projection. The restriction of \( h_E \) to \( E \cap [W \times \mathbb{A}^1 \times \mathbb{A}^1] \) gives an isomorphism

\[
E \cap [W \times \mathbb{A}^1 \times \mathbb{A}^1] \to W \times \mathbb{A}^1,
\]
which thus defines the section

\[ s_E: W \times \mathbb{A}^1 \rightarrow E \]

to \( h_E \).

The isomorphisms (2), (3) and (4) give us inclusions

\[
i_{*0}: Y_W \rightarrow Q; \ i_{0*}: Y_Z \rightarrow Q; \ i_{*1}: Y_T \rightarrow Q; \ i_{1*}: E \rightarrow Q; \\
i_{01}: X \rightarrow Q; \ i_{10}: P_W \rightarrow Q; \ i_{01}: P_Z \rightarrow Q; \ i_{11}: P_T \rightarrow Q; \quad (5) \\
i_{E0}: P_W \rightarrow E; \ i_{E1}: P_T \rightarrow E.
\]

We use the convention that the image of \( i_{ab} \) lies in the fiber over \( X \times (a, b) \), that of \( i_{*b} \) lies in the fiber over \( X \times \mathbb{A}^1 \times b \), etc.

Putting (2)-(5) together gives the commutative diagram of isomorphisms

\[
\begin{array}{c}
\mathbb{Z}_{X, \hat{W}} & \xleftarrow{i_{*0}} & \mathbb{Z}_{Y_Z, \hat{Y}_Z} & \xrightarrow{i_{*Z}} & \mathbb{Z}_{P_Z, \hat{P}_Z} \\
\uparrow i_{*0} & \swarrow i_{0*} & \uparrow i_{01} & \swarrow i_{*0} & \uparrow i_{*0} \\
\mathbb{Z}_{Y_W, \hat{Y}_W} & \xleftarrow{i_{*0}} & \mathbb{Z}_{Q, \hat{Q}} & \xrightarrow{i_{*1}} & \mathbb{Z}_{Y_T, \hat{Y}_T} \\
\downarrow i_{W1} & \swarrow i_{10} & \downarrow i_{11} & \searrow i_{*1} & \downarrow i_{T1} \\
\mathbb{Z}_{P_W, \hat{P}_W} & \xleftarrow{i_{*E0}} & \mathbb{Z}_{E, \hat{E}} & \xrightarrow{i_{*E1}} & \mathbb{Z}_{P_T, \hat{P}_T}
\end{array}
\]

Let \( i_0: W \rightarrow W \times \mathbb{A}^1, \ i_1: W \rightarrow W \times \mathbb{A}^1 \) be the inclusions \( i_0(w) = (w, 0), \ i_1(w) = (w, 1) \).

We have the commutative diagram of isomorphisms

\[
\begin{array}{c}
\mathbb{Z}_{W, \hat{W}}(-d - c)[-2d - 2e] & \xleftarrow{s_{\bar{W}}^{\bar{W}}} & \mathbb{Z}_{T, \hat{T}}(-d - c)[-2d - 2e] & \xrightarrow{\cup[s_T(W)] \circ f_{T}} & \mathbb{Z}_{P_T, \hat{P}_T} \\
\uparrow i_{*}^{*} & & & \uparrow i_{*E1} \\
\mathbb{Z}_{W \times \mathbb{A}^1, \hat{W} \times \mathbb{A}^1}(-d - c)[-2d - 2e] & \xrightarrow{\cup[s_{W}(W \times \mathbb{A}^1)] \circ h_{E}} & \mathbb{Z}_{E, \hat{E}} & \xrightarrow{\downarrow i_{*E0}} & \mathbb{Z}_{P_W, \hat{P}_W} \\
\downarrow i_{0}^* & & & & \end{array}
\]

Since the composition \( (i_{0}^{*})^{-1} \circ i_{*}^{*} \) is the identity, the composition

\[
\begin{array}{c}
\mathbb{Z}_{W, \hat{W}}(-d - c)[-2d - 2e] & \xrightarrow{\cup[s_{W}(W)] \circ f_{W}} & \mathbb{Z}_{P_W, \hat{P}_W} & \xrightarrow{(i_{E0}^{*})^{-1}} & \mathbb{Z}_{E, \hat{E}} & \xrightarrow{i_{*E1}} & \mathbb{Z}_{P_T, \hat{P}_T} \\
\xrightarrow{(\cup[s_T(W)] \circ f_{T})^{-1}} & & & & \mathbb{Z}_{T, \hat{T}}(-d - c)[-2d - 2e] & \xrightarrow{s_{\bar{W}}^{\bar{W}}} & \mathbb{Z}_{W, \hat{W}}(-d - c)[-2d - 2e]
\end{array}
\]

is the identity.
The maps $i_*$, $(i \circ j)_*$, $j_*$ and $i_{T*}$ are defined by the respective compositions

$$i_* : Z_\mathcal{W}(-d)[-2d] \xrightarrow{\cup[s_Z(Z)] \circ f_Z^*} Z_{P_\mathcal{Z}, \mathcal{P}} \xrightarrow{(i_Z^*)^{-1}} Z_{Y_\mathcal{W}, \mathcal{Y}^\mathcal{W}} \xrightarrow{i_{\mathcal{Y}W}^{-1}} Z_X, \mathcal{W};$$

$$(i \circ j)_* : Z_{\mathcal{W}, \mathcal{W}}(-d - e)[-2d - 2e] \xrightarrow{\cup[s_{\mathcal{W}}(\mathcal{W})] \circ f_W^*} Z_{P_\mathcal{W}, \mathcal{P}_W} \xrightarrow{(i_{W1}^{-1})} Z_{Y_W, \mathcal{Y}_W} \xrightarrow{i_{\mathcal{Y}W}^{-1}} Z_X, \mathcal{W};$$

$$j_* : Z_{\mathcal{W}, \mathcal{W}}(-e)[-2e] \xrightarrow{\cup[s_{\mathcal{W}}(\mathcal{W})] \circ f_W^*} Z_{P_\mathcal{W}, \mathcal{P}_W} \xrightarrow{(i_{W}^{-1})^{-1}} Z_{Y_W, \mathcal{Y}_W} \xrightarrow{i_{\mathcal{Y}W}^{-1}} Z_X, \mathcal{W};$$

$$i_{T*} : Z_{T, \mathcal{T}}(-d - e)[-2d - 2e] \xrightarrow{\cup[s_{T}(\mathcal{T})] \circ f_{T}^*} Z_{P_\mathcal{T}, \mathcal{P}_T} \xrightarrow{(i_{T1}^{-1})} Z_{Y_T, \mathcal{Y}_T} \xrightarrow{i_{\mathcal{Y}T}^{-1}} Z_{P_\mathcal{T}, \mathcal{T}}.$$

Combining this with the diagrams (1) and (7), and the definition (8) of the maps $i_*$, $j_*$, $(i \circ j)_*$ and $i_{T*}$ completes the proof.

(6.2.2) Proposition (projection formula)

Let $i : Z \to X$ be a closed embedding in $\mathcal{V}$ of codimension $d$, $\mathcal{Z}_1$, $\mathcal{Z}_2$ closed subsets of $Z$, and $\mathcal{X}_1$ closed subsets of $X$ containing $\mathcal{Z}_i$, $i = 1, 2$. Then the diagram

$$Z_{\mathcal{Z}_1, \mathcal{Z}_2}(-d)[-2d] \otimes Z_{\mathcal{X}_1, \mathcal{X}_2} \xrightarrow{i_* \otimes \text{id}} Z_{\mathcal{Z}_1, \mathcal{Z}_2}(-d)[-2d] \otimes Z_{\mathcal{Z}_2, \mathcal{Z}_2} \xrightarrow{\cup \mathcal{Z}} Z_{\mathcal{Z}_1 \cap \mathcal{Z}_2}(-d)[-2d] \xrightarrow{i_*} Z_{\mathcal{X}_1 \cap \mathcal{X}_2, \mathcal{X}_2}$$

commutes in $\mathcal{D}_M(S)$.

Proof. For ease of notation, we give the proof in the case $Z = \mathcal{Z}_1 = \mathcal{Z}_2$, $X = \mathcal{X}_1 = \mathcal{X}_2$. Via the diagram (6.1.1.1), we have the definition of the map

$$i_* : Z_S(-d)[-2d] \to Z_X$$

as the composition

$$Z_S(-d)[-2d] \xrightarrow{f_*} Z_{P}(-d)[-2d] \xrightarrow{\cup[s(Z)]} Z_{P, S(Z)} \xrightarrow{(i_*^{-1})^{-1}} Z_{Y, Z \mathcal{X} \mathcal{A}_1} \xrightarrow{i_0^{-1}} Z_X.$$
Taking the product of (6.1.1.1) with \( X \) yields the diagram

\[
\begin{array}{cccccc}
X_X & \xrightarrow{i_{X_0}} & Y_X & \longleftarrow & \xrightarrow{i_{X_1}} & P_X \\
\| & & q_X \downarrow \ & & & \downarrow s_X \\
X_X \times 0 & \xrightarrow{id_X \times j_0} & X_X \times A^1 & \longleftarrow & Z_X \times A^1 & \longleftarrow & Z_X = Z_X \times 1
\end{array}
\] (2)

which gives the definition of the map

\[(i \times \text{id}_X)_*: \mathbb{Z}_{Z \times X}(-2d) \to \mathbb{Z}_{X \times X, Z \times X}\]

as the composition

\[
\mathbb{Z}_{Z \times X}(-d)[-2d] \xrightarrow{f^*} \mathbb{Z}_{P_X}(-d)[-2d] \xrightarrow{\cup [s_X(Z_X)]} \mathbb{Z}_{P_X, s_X(Z_X)} \xrightarrow{(i_{X_1})^{-1}} \mathbb{Z}_{Y_X, Z \times X, A^1} \xrightarrow{i_{X_0}^\circ} \mathbb{Z}_{X \times X}. \quad (3)
\]

We have the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}_Z(-d)[-2d] \otimes \mathbb{Z}_X & \xrightarrow{\Delta Z \times X} & \mathbb{Z}_{Z \times X}(-d)[-2d] \\
\downarrow i_* \otimes \text{id} & & \downarrow i_* \\
\mathbb{Z}_X \otimes \mathbb{Z}_X & \xrightarrow{\Delta_{X \times X}} & \mathbb{Z}_{X \times X},
\end{array}
\]

which, together with (1) and (3), yields the commutativity of the diagram

\[
\begin{array}{ccc}
\mathbb{Z}_Z(-d)[-2d] \otimes \mathbb{Z}_X & \xrightarrow{\Delta Z \times X} & \mathbb{Z}_{Z \times X}(-d)[-2d] \\
\downarrow i_* \otimes \text{id} & & \downarrow i_* \\
\mathbb{Z}_X \otimes \mathbb{Z}_X & \xrightarrow{\Delta_{X \times X}} & \mathbb{Z}_{X \times X},
\end{array}
\]

The naturality of the external products \( \Box_{\ast, \ast} \) implies that the diagram

\[
\begin{array}{ccc}
\mathbb{Z}_Z(-d)[-2d] \otimes \mathbb{Z}_X & \xrightarrow{\text{id}_Z \otimes i_*} & \mathbb{Z}_Z(-d)[-2d] \otimes \mathbb{Z}(m) \\
\downarrow \Box Z \times X & & \downarrow \Box Z \times Z \\
\mathbb{Z}_{Z \times X}(-d)[-2d] & \xrightarrow{(\text{id}_Z \times i)_*} & \mathbb{Z}_{Z \times X}(-d)[-2d]
\end{array}
\]

commutes. Thus, we need only check the commutativity of the diagram

\[
\begin{array}{ccc}
\mathbb{Z}_{Z \times X}(-d)[-2d] & \xrightarrow{(\text{id}_Z \times i)_*} & \mathbb{Z}_Z(-d)[-2d] \\
\downarrow (i \times \text{id}_X)_* & & \downarrow i_* \\
\mathbb{Z}_{X \times X} & \xrightarrow{\Delta_X} & \mathbb{Z}_X
\end{array}
\] (4)
We have the commutative diagram
\[
\begin{array}{c}
Z \xrightarrow{(i, \text{id}_Z)} Z_X = Z \times X \\
i \downarrow \quad \quad \quad \downarrow \text{id}_X \\
X \xrightarrow{\Delta_x} X_X = X \times X.
\end{array}
\]

We take the product with $\mathbb{A}^1$, and blow-up along $Z \times 1$ and $Z \times X \times 1$, which, together with the diagram (6.1.1.1) and (2), gives us the commutative diagram
\[
\begin{array}{c}
Z \xrightarrow{s} P \xrightarrow{i_1} Y \xleftarrow{i_0} X \\
\xrightarrow{(i, \text{id}_Z)} \quad \downarrow \delta_p \quad \downarrow \delta_Y \quad \downarrow \Delta_x \\
Z_X \xrightarrow{f_X} P_X \xrightarrow{i_{X_1}} Y_X \xleftarrow{i_{X_0}} X_X
\end{array}
\]

In addition, we have
\[
\delta^*_Y([Z_X \times \mathbb{A}^1]) = [Z \times \mathbb{A}^1]; \quad \delta^*_p(s_X([Z_X])) = s([Z]). \tag{6}
\]

Putting (5) and (6) together gives the commutative diagram
\[
\begin{array}{c}
\mathbb{Z}_{X_X} \xleftarrow{i^*_{X_0}} \quad \Delta^*_X \quad \mathbb{Z}_X \\
\mathbb{Z}_{Y_X, Z_X \times \mathbb{A}^1} \quad \delta^*_Y \quad \mathbb{Z}_{Y, Z_X \times \mathbb{A}^1} \\
i_{X_1} \downarrow \quad \downarrow \quad \downarrow \delta_Y \\
\mathbb{Z}_{P_X, s_X(Z_X)} \quad \delta_p \quad \mathbb{Z}_{P, s(Z)} \\
\uparrow[s_X(Z_X)] \quad \uparrow \delta_p \quad \uparrow[s(z)] \\
\mathbb{Z}_{P_X}(-d)[-2d] \quad \delta^*_p \quad \mathbb{Z}_P(-d)[-2d] \\
\mathbb{Z}_{Z \times X}(-d)[-2d] \quad (i, \text{id}_Z)^* \quad \mathbb{Z}_Z(-d)[-2d]
\end{array}
\]

As this implies the commutativity of (4), the proof is complete.

We use the notation $\text{cl}_{\mathbb{A}, \mathbb{A}}^k(B)$ to denote the cycle class in $H^{2k}_{\mathbb{A}}(A, \mathbb{Z}(k))$ of a codimension $k$ cycle $B$ on $A$, supported on $\mathbb{A}$.

(6.2.3) Theorem

Let $i: Z \to X$ be a closed embedding in $\text{Sm}_S$, of codimension $d$, and let $W$ be in $\mathbb{Z}^p(Z/S)$, supported on a closed subset $\mathcal{Z}$ of $Z$. Then
\[
i_*(\text{cl}^p_{Z, \mathcal{Z}}(W)) = \text{cl}^{p+q}_{X, \mathcal{Z}}(i_*(W)).
\]
Proof. We use the notation from (6.1.1). From (3.3.3), we have

\[
(\cup [s(Z)] \circ f^*)(\text{cl}_{X}^P Z, W)) = \text{cl}_{P, s(Z)}^P(s(Z) \cup (f^*(W)))
\]

\[
= \text{cl}_{P, s(Z)}^P(s(W)).
\]

\[
i_1^*(\text{cl}_{Y, Z \times A^1}^P(i_*(W) \times A^1)) = \text{cl}_{P, s(Z)}^P(s(W)).
\]

\[
i_0^*(\text{cl}_{Y, Z \times A^1}^P(i_*(W) \times A^1)) = \text{cl}_{X, Z}^P(i_*(W)).
\]

These identities, together with the definition of \(i_\ast\), proves the lemma.

(6.2.4) **Lemma**

Let \(i : Z \to X\) be a closed embedding in \(\text{Sm}_S\), \(p : W \to X\) a morphism \(\text{Sm}_S\), giving the cartesian diagram

\[
\begin{array}{ccc}
W \times_X Z & \overset{p_1}{\longrightarrow} & W \\
p_2 \downarrow & & \downarrow p \\
Z & \overset{i}{\longrightarrow} & X
\end{array}
\]

Suppose that \(W \times_X Z\) is in \(\text{Sm}_S\). Then

\[
p^* \circ i_\ast = p_1^* \circ p_2^*.
\]

**Proof.** We use the notation from (6.1.1), and give the proof without closed supports to simplify the notation.

Applying the product \(W \times_X (-)\) to the diagram (6.1.1.1) gives the diagram

\[
\begin{array}{cccc}
W & \overset{iw_0}{\longrightarrow} & W \times_X Y & \overset{iw_1}{\longleftarrow} & W \times_X P \\
\| & & \| & & \| \overset{s_{w}}{\downarrow} \overset{q_{w}}{\downarrow} \overset{f_{w}}{\downarrow} \\
W \times 0 & \overset{id_{W} \times j_0}{\longrightarrow} & W \times A^1 & \overset{id_{W} \times id_{A^1}}{\longleftarrow} & W \times_X Z \times A^1 & \overset{id_{W} \times j_1}{\longleftarrow} & W \times_X Z \times 1
\end{array}
\]

We have the commutative diagram

\[
\begin{array}{cccc}
W \times_X Z & \overset{sw}{\longrightarrow} & W \times_X P & \overset{iw_1}{\longleftarrow} & W \times_X Y & \overset{iw_0}{\longleftarrow} & W \\
\| & p_2 \downarrow & p_2 \downarrow & p_2 \downarrow & \| & \downarrow p & \\
Z & \overset{s}{\longrightarrow} & P & \overset{i_1}{\longrightarrow} & Y & \overset{i_0}{\longleftarrow} & X
\end{array}
\]

In addition, we have

\[
p_2^*([Z \times A^1]) = [W \times_X Z \times A^1]; \quad p_2^*(s(Z)) = s_{W}(W \times_X Z).
\]
This gives us the commutative diagram

\[
\begin{array}{ccc}
Z_{W, W \times \mathbb{A}^1} & \xrightarrow{p^*} & Z_{X, \mathbb{A}^1} \\
\uparrow i_{W_0}^* & & \uparrow i_0^* \\
Z_{W \times X, Y, W \times X \times \mathbb{A}^1} & \xrightarrow{p_2^*} & Z_{Y, Z \times \mathbb{A}^1} \\
\downarrow i_{W_1}^* & & \downarrow i_1^* \\
Z_{W \times X, P, s(W \times X Z)} & \xrightarrow{p_2^*} & Z_{P, s(Z)} \\
\uparrow \cup [s_{W(W \times X Z)}] & & \uparrow \cup [s(Z)] \\
Z_{W \times X, P, s(W \times X Z)}(-d)[-2d] & \xrightarrow{p_2^*} & Z_{P, s(Z)}(-d)[-2d] \\
\uparrow f^*_W & & \uparrow f^* \\
Z_{W \times X Z}(-d)[-2d] & \xrightarrow{p_2^*} & Z_{Z}(-d)[-2d]
\end{array}
\]

(1)

By definition, \( p_{1*} \) is the composition

\[
Z_{W \times X Z}(-d)[-2d] \xrightarrow{\cup [s_{W(W \times X Z)}] \circ f^*_W} Z_{W \times X, P, s(W \times X Z)} \xrightarrow{(i_{W_1}^*)^{-1}} Z_{W \times X, Y, W \times X Z \times \mathbb{A}^1} \xrightarrow{i_{W_0}^*} Z_{W, W \times X Z}.
\]

This, together with the definition of \( i_* \) and the diagram (1), completes the proof. \qed

(6.2.5) Theorem (Purity)

Suppose that the base scheme \( S \) is \( \text{Spec}(k) \) for a field \( k \). In case \( \text{char}(k) = p > 0 \), we suppose further that the coefficient ring \( R \) is \( \mathbb{Q} \). Let \( X \) be in \( \text{Sm}_S \), and \( \hat{X} \) a closed subset of codimension \( \geq d \). Then

\[
H^{2d-p}_X(X, R(q)) = 0 \text{ if } p > 0 \text{ and } q = d, \text{ or if } q < d.
\]

Proof. We proceed by induction on \( q \). Let \( \hat{Y} \subset \hat{X} \) be the singular locus of \( \hat{X} \), together with all components of \( \hat{X} \) which have codimension \( > d \), so

\[
\text{codim}_X(\hat{Y}) \geq d + 1.
\]

We have the distinguished triangle (2.2.9.2)

\[
Z_{X, \hat{Y}}(q) \rightarrow Z_{X, \hat{X}}(q) \rightarrow Z_{X \setminus \hat{Y}, \hat{X} \setminus \hat{Y}} \rightarrow Z_{X, \hat{Y}}(q)[1];
\]

Applying the induction hypothesis to the long exact cohomology sequence associated to this triangle, we reduce to the case of a smooth \( \hat{X} \), of pure codimension \( q \).
In this case, we have the isomorphism

\[ i_*: \mathbb{Z}_X(q - d) \to \mathbb{Z}_{X, \hat{X}}(q)[2d], \]

giving the isomorphism

\[ i_*: H^{-p}(\hat{X}, R(q - d)) \to H_{2q-p}^{2}(X, R(q)). \]

By (4.6.6), we have

\[ H^{-p}(\hat{X}, R(q - d)) = \text{CH}^{-d}(\hat{X}, 2(q - d) + p) \otimes R \]

which is zero if \( p > 0 \) and \( q = d \), or if \( q < d \). Thus, the induction goes through, completing the proof. \( \square \)

We now check that, in case of a section \( i: Z \to X \) to a smooth projection \( q: X \to Z \), the definition (6.1.1.2) of \( i_* \) agrees with the Gysin morphism \( \cup [i(Z)] \circ i^* \) of (2.2.5).

### (6.2.6) Lemma

Let \( q: X \to Z \) be a morphism in \( \text{Sm}_S \), with section \( i: Z \to X \), \( \hat{Z} \) a closed subset of \( Z \), and \( \hat{X} \) a closed subset of \( X \) containing \( s(\hat{Z}) \). Let \( d = \text{codim}_X(i(Z)) \). Then the two maps \( \cup [i(Z)] \circ i^* \) and \( i_* \), from \( \mathbb{Z}_{Z, \hat{Z}}(-d)[-2d] \) to \( \mathbb{Z}_{X, \hat{X}} \), agree in \( DM(S) \).

**Proof.** We use the notation of (6.1.1). The map \( q \) induces maps \( q_Y: Y \to Z \times \mathbb{A}^1 \), \( q_P: P \to Z \), with \( s' \) a section to \( q_Y \), and \( q_P = f \). Letting \( i_{Z0}, i_{Z1}: Z \to Z \times \mathbb{A}^1 \) be the 0 and 1 sections, respectively, we have the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_0} & Y \\
q \downarrow i & & q_Y \downarrow s' \\
Z & \xrightarrow{i_{Z0}} & Z \times \mathbb{A}^1 \\
& i_{Z1} & \leftarrow \end{array}
\]

In addition, we have the identity of cycles

\[ i(Z) = i_0^*(s'(Z \times \mathbb{A}^1)). \]

This gives the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}_{Z, \hat{Z}} & \xrightarrow{\cup [i(Z)] \circ q^*} & \mathbb{Z}_{X, \hat{X}} \\
i_{Z0}^* & \uparrow & i_0^* \\
\mathbb{Z}_{Z \times \mathbb{A}^1, \hat{Z} \times \mathbb{A}^1} & \xrightarrow{\cup [s'(Z \times \mathbb{A}^1)] \circ q_Y^*} & \mathbb{Z}_{Y, s'(\hat{Z} \times \mathbb{A}^1)} \\
i_{Z1}^* & \downarrow & i_1^* \\
\mathbb{Z}_{Z, \hat{Z}} & \xrightarrow{\cup [s(Z)] \circ f^*} & \mathbb{Z}_{P, s(\hat{Z})} \\
\end{array}
\] (1)

As \( i_{Z0}^* = (p_1^*)^{-1} = i_{Z1}^* \), (1), together with the definition of \( i_* \), completes the proof. \( \square \)

Finally, we check the compatibility of the Gysin morphism with the external products:
**Proposition**

Let \( i : Z \to X \) be a closed embedding in \( \text{Sm}_S \) of codimension \( d \), \( \hat{Z} \) a closed subset of \( Z \) and \( \hat{X} \) a closed subset of \( X \) with \( \hat{Z} \subset \hat{X} \). Let \( W \) be in \( \text{Sm}_S \), \( \hat{W} \) a closed subset of \( W \). Then the diagram

\[
\begin{array}{ccc}
Z_{Z,\hat{Z}}(-d)[-2d] \otimes Z_{\hat{W},\hat{W}} & \xrightarrow{i_* \otimes \text{id}} & Z_{X,\hat{X}} \otimes Z_{W,\hat{W}} \\
\downarrow & & \downarrow \\
\mathcal{D}(S) = \hat{Z} \times _S W & \to & \hat{X} \times _S W \\
\end{array}
\]

commutes in \( \mathcal{DM}(S) \).

**Proof.** Taking the diagram (6.1.1.1) and forming the product \((-) \times _S W\) gives the diagram defining the map \((i \times \text{id}_W)_*\). We have the commutative diagram

\[
\begin{array}{ccc}
Z_{X,\hat{X}} \otimes Z_{W,\hat{W}} & \xrightarrow{\text{id}_X \otimes \text{id}_W} & Z_{X \times _S W, \hat{X} \times _S \hat{W}} \\
\downarrow & & \downarrow \\
Z_{\hat{X},\hat{W}} \otimes Z_{W,\hat{W}} & \xrightarrow{\text{id}_X \otimes \text{id}_W} & Z_{X \times _S W, \hat{X} \times _S \hat{W}} \\
\downarrow & & \downarrow \\
Z_{\hat{X},\hat{W}} \otimes Z_{W,\hat{W}} & \xrightarrow{\text{id}_X \otimes \text{id}_W} & Z_{X \times _S W, \hat{X} \times _S \hat{W}} \\
\downarrow & & \downarrow \\
Z_{\hat{X},\hat{W}} \otimes Z_{W,\hat{W}} & \xrightarrow{\text{id}_X \otimes \text{id}_W} & Z_{X \times _S W, \hat{X} \times _S \hat{W}} \\
\end{array}
\]

This, together with the definition of \( i_* \) and \((i \times \text{id}_W)_*\), completes the proof. \(\square\)
6.3. Push-forward for a projection

We use the projective bundle formula to define the push-forward $q_*$ for

$$q: \mathbb{P}(E) \to X$$

the projective space bundle associated to a vector bundle $E \to X$.

(6.3.1) The definition of push-forward for a projection

Let $p: E \to X$ be a vector bundle of rank $N + 1$, $q: \mathbb{P}(E) \to X$ the associated $\mathbb{P}^N$-bundle with tautological bundle $\mathcal{O}_E(1)$. Let $\hat{X}$ be a closed subset of $X$, and $\hat{P}_E$ the inverse image $q^{-1}(\hat{X})$. We let $\zeta = \text{cl}^1(\mathcal{O}(1))$. By (5.4.2), we have the isomorphism

$$\alpha_{X,\hat{X}}^E := \sum_{i=0}^N \alpha_i^E \cdot \oplus_{i=0}^N \mathbb{Z}_{X,\hat{X}}(N - i)[2N - 2i] \to \mathbb{Z}_{\mathbb{P}(E),\hat{P}}(2N).$$

We define the map

$$q_*: \mathbb{Z}_{\mathbb{P}(E),\hat{P}}(N)[2N] \to \mathbb{Z}_{X,\hat{X}}$$

(6.3.1.1)

to be the composition $\pi_N \circ (\alpha_{X,\hat{X}}^E)^{-1}$, where

$$\pi_N: \oplus_{i=0}^N \mathbb{Z}_{X,\hat{X}}(N - i)[2N - 2i] \to \mathbb{Z}_{X,\hat{X}}$$

is the projection on the summand $i = N$.

Suppose we have a surjection of vector bundles on $X$: $j: E \to F$, which thus induces the closed embedding over $X$

$$\tilde{j}: \mathbb{P}(F) \to \mathbb{P}(E)$$

of codimension, say, $d$. Let $q_E: \mathbb{P}(E) \to X$, $q_F: \mathbb{P}(F) \to X$ be the structure morphisms.

(6.3.2) Lemma

Let $F$ be the trivial rank $N - d + 1$ bundle $1^{N-d+1}$ on $X$, $F'$ the trivial rank $d$ bundle $1^d$, and let $E$ be the trivial rank $N + 1$ bundle $F \oplus F'$. Let $j: E \to F$ be the projection with kernel $F'$. Then we have

$$q_{F*} = q_{E*} \circ j_*.$$

Proof. Let $\zeta_E = c_1(\mathcal{O}_E(1))$, $\zeta_F = c_1(\mathcal{O}_F(1))$. By the naturality of the first Chern class (5.3.3)(i), and of the tautological bundle $\mathcal{O}(1)$, we have

$$j^*(\zeta_E) = \zeta_F.$$  \hspace{1cm} (1)

Let $\hat{X}$ be a closed subset of $X$, with inverse images $\hat{P}_F$ in $\mathbb{P}(F)$, and $\hat{P}_E$ in $\mathbb{P}(E)$. Let

$$\alpha_i^E: \mathbb{Z}_{X,\hat{X}}(-i)[-2i] \to \mathbb{Z}_{\mathbb{P}(E),\hat{P}_E}$$

$$\alpha_i^F: \mathbb{Z}_{X,\hat{X}}(-i)[-2i] \to \mathbb{Z}_{\mathbb{P}(F),\hat{P}_F}$$
be the maps (5.4.1.1), i.e., the respective compositions

\[ Z_{X,E}(-i)[-2i] q_F^* Z_{P(E), \hat{P}_E}(-i)[-2i] Z_{P(E), \hat{P}_E} \]

Let \(|\mathbb{P}(F)|\) denote the cycle on \(\mathbb{P}(E)\) determined by the subscheme \(\mathbb{P}(F)\); by the Appendix, (A.4.9), \(|\mathbb{P}(F)|\) is an element of \(\mathbb{Z}(\mathbb{P}(E)/S)\). Since \(q_F = q_E \circ j\),

\[ \tilde{j}_* \circ \tilde{j}^* \circ q_E^* = \tilde{j}_* \circ \tilde{j}^* \circ ((-) \cup \text{cl}^0_{\mathbb{P}(E)}(|\mathbb{P}(E)|) \circ q_E^*) \quad (3.3.6) \]

\[ = \tilde{j}_* \circ ((-) \cup \text{cl}^0_{\mathbb{P}(F)}(|\mathbb{P}(F)|) \circ (\tilde{j}^* \circ q_E^*)) \quad (3.3.3) \]

\[ = \cup \text{cl}^d_{\mathbb{P}(E)}(|\mathbb{P}(F)|) \circ q_E^* \quad (6.2.2) \quad (2) \]

It follows from (1), (2) and another application of the projection formula (6.2.2) that the diagram

\[ Z_{X,E}(-i-d)[-2i-2d] \xrightarrow{\alpha_i^E} Z_{P(E), \hat{P}_E}(-d)[-2d] \]

\[ \quad \downarrow \tilde{j}_* \]

\[ Z_{P(E), \hat{P}_E}(-d)[-2d] \xrightarrow{\cup \text{cl}^d(|\mathbb{P}(F)|)} Z_{P(E), \hat{P}_E} \]

is commutative.

On the other hand, let \(H_i\) be the hyperplane of \(\mathbb{P}(E)\) corresponding to the projection \(E \to 1^N\) omitting the \(i\)th summand. Then \(H_i\) is the divisor of a section of \(\mathcal{O}_E(1)\) and is in \(\mathbb{Z}^1(\mathbb{P}(E)/S)\), hence, by (5.3.3)(iii), we have

\[ \text{cl}^1(H_i) = \zeta_E. \]

Thus

\[ \text{cl}^d(|\mathbb{P}(F)|) = \text{cl}^d(H_{i_1} \cap \ldots \cap H_{i_d}) \]

\[ = \text{cl}^1(H_{i_1}) \cup \ldots \cup \text{cl}^1(H_{i_d}) \]

\[ = \zeta^d_E. \quad (4) \]

Combining (4) with the diagram (3), we have the identity

\[ \alpha_i^E = \tilde{j}_* \circ \alpha_{i-d}^F, \text{ for } d \leq i < N. \]

This implies the desired identity \(q_{F*} = q_{E*} \circ \tilde{j}_*\).

\(\blacksquare\)

(6.3.3) **Lemma**

Let \(p: E \to X\) be a rank \(N+1\) vector bundle, \(L \to X\) a line bundle, \(p': E' \to X\) the vector bundle \(E \times_X L\), \(q: \mathbb{P}(E) \to X\), \(q': \mathbb{P}(E') \to X\) the respective \(\mathbb{P}^N\)-bundles, and
\( \rho: \mathbb{P}(E) \to \mathbb{P}(E') \) the canonical isomorphism. Let \( \hat{X} \) be a closed subset of \( X \), \( \hat{P}_E = q^{-1}(\hat{X}) \), \( \hat{P}_{E'} = q'^{-1}(\hat{X}) \). Then the diagram

\[
\begin{array}{c}
\mathbb{Z}_{\mathbb{P}(E'), \hat{P}_{E'}}(N)[2N] \quad \rho^* \quad \mathbb{Z}_{\mathbb{P}(E'), \hat{P}_{E'}}(N)[2N] \\
\downarrow q_* \quad \downarrow q_* \\
\mathbb{Z}_{X, \hat{X}} \quad = \quad \mathbb{Z}_{X, \hat{X}}
\end{array}
\]

commutes.

**Proof.** Let \( \zeta_E = c_1(\mathcal{O}_E(1)) \), \( \zeta_{E'} = c_1(\mathcal{O}_{E'}(1)) \). Then

\[
\rho^*(\zeta_{E'}) = \zeta_E + q^*(L),
\]

hence

\[
\rho^*(\zeta_{i, E'}) = \zeta_E^i + \sum_{j=0}^{i-1} \zeta_E^j q^*(x_i),
\]

for suitable \( x_i \in H^{2i-2j}(X, \mathbb{Z}(i-j)) \). Let

\[
\alpha^E: \bigoplus_{i=0}^N \mathbb{Z}_{X, \hat{X}}(-i)[-2i] \to \mathbb{Z}_{\mathbb{P}(E), \hat{P}_E}
\]

\[
\alpha^{E'}: \bigoplus_{i=0}^N \mathbb{Z}_{X, \hat{X}}(-i)[-2i] \to \mathbb{Z}_{\mathbb{P}(E'), \hat{P}_{E'}}
\]

be the isomorphisms given by the projective bundle formula (5.4.2). By (1), there is an upper triangular matrix \( M \), with one’s along the diagonal, and with entries in \( H^*(X, \mathbb{Z}(*)) \), such that the diagram

\[
\begin{array}{c}
\mathbb{Z}_{\mathbb{P}(E'), \hat{P}_{E'}}(N)[2n] \quad \rho^* \quad \mathbb{Z}_{\mathbb{P}(E'), \hat{P}_{E'}}(N)[2n] \\
\uparrow \alpha^{E'} \quad \uparrow \alpha^E \\
\bigoplus_{i=0}^N \mathbb{Z}_{X, \hat{X}}(-i)[-2i] \quad \to \quad \bigoplus_{i=0}^N \mathbb{Z}_{X, \hat{X}}(-i)[-2i]
\end{array}
\]

commutes. This, together with the definition of \( q_* \) and \( q'_* \), completes the proof.  

\[\square\]

**6.3.4 Proposition (Projection Formula)**

Let \( p: E \to X \) be a vector bundle of rank \( N + 1 \), \( q: P = \mathbb{P}(E) \to X \) the associated \( \mathbb{P}^N \)-bundle, \( \hat{X}_i, i = 1, 2 \) closed subsets of \( X \), and \( \hat{P}_i = q^{-1}(\hat{X}_i) \). Then the diagram

\[
\begin{array}{c}
\mathbb{Z}_{P, \hat{P}_1}(N)[2N] \otimes \mathbb{Z}_{X, \hat{X}_2} \quad \xrightarrow{id_P \otimes q^*} \quad \mathbb{Z}_{P, \hat{P}_1}(N)[2N] \otimes \mathbb{Z}_{P, \hat{P}_2} \\
\downarrow q_* \otimes \text{id} \quad \downarrow \cup_P \\
\mathbb{Z}_{X, \hat{X}_1} \otimes \mathbb{Z}_{X, \hat{X}_2} \quad \xrightarrow{\cup_X} \quad \mathbb{Z}_{X, \hat{X}_1 \cap \hat{X}_2}
\end{array}
\]
commutes.

**Proof.** We give the proof in case $\hat{X}_1 = \hat{X}_2 = X$ to simplify the notation. The associativity and graded-commutativity of products implies the commutativity of the diagram

$$
\begin{array}{ccc}
Z_X \otimes Z_X & \overset{\alpha_Z^N \otimes q^*}{\longrightarrow} & Z_{P, P_1}(N)[2N] \otimes Z_{P, P_2} \\
\cup_X & \searrow & \downarrow_{\cup_P} \\
Z_X & \longrightarrow & Z_P(N)[2N]
\end{array}
$$

This, together with the definition of $q_*$, proves the proposition. □

**Lemma (6.3.5)**

Let $q: \mathbb{P}(E) \to X$ be the projective bundle associated to a rank $N + 1$ vector bundle $E \to X$ on $X$, $p: W \to X$ a morphism in $\text{Sm}_S$, giving the cartesian diagram

$$
\begin{array}{ccc}
\mathbb{P}(p^* E) = W \times_X \mathbb{P}(E) & \overset{p_2}{\longrightarrow} & \mathbb{P}(E) \\
p_1 \downarrow & & \downarrow q \\
W & \longrightarrow & X
\end{array}
$$

Then

$$p^* \circ q_* = p_1^* \circ p_2^*.$$

**Proof.** Let $\zeta = c_1 O_E(1)$, $\zeta_W = c_1 O_{p^* E}(1)$. Then

$$p_2^* (\zeta) = \zeta_W.$$

This implies the relation

$$p_2^* \circ \alpha_i^E = \alpha_i^{p^* E} \circ p^*,$$

which in turn implies the desired result. □

**Remark (6.3.6)**

In the case of the dimension 0 projective bundle, $q: \mathbb{P}^0_X = X \to X$, the projective bundle isomorphism is the identity map, hence $q_* = \text{id}$. □

**Proposition (6.3.7)**

Let $p: E \to X$ be a vector bundle of rank $N + 1$, $q: P = \mathbb{P}(E) \to X$ the associated $\mathbb{P}^N$-bundle, $\hat{X}_1$, a closed subset of $X$, $\hat{P} = q^{-1}(\hat{X})$. Let $W$ be in $\text{Sm}_S$, $\hat{W}$ a closed subset of $W$; this gives us the projective bundle $q \times \text{id}_W: P \times_S W \to X \times_S W$ associated to the vector bundle $p^*_1 E \to X \times_S W$. Then the diagram

$$
\begin{array}{ccc}
Z_{P, P}(N)[2N] \otimes Z_{W, W} & \overset{q \otimes \text{id}}{\longrightarrow} & Z_{X, X} \otimes Z_{W, W} \\
\circ_{P, W} \downarrow & & \downarrow \circ_{X, W} \\
Z_{P \times_S W, P \times_S W}(N)[2N] & \overset{\text{(q \times \text{id}_W)}_*}{\longrightarrow} & Z_{X \times_S W, X \times_S W}
\end{array}
$$
commutes.

Proof. Since
\[ p_1^*(c_1(O_E(1))) = c_1(O_{p_1^*E}(1)), \]
we have the commutative diagram
\[
\begin{array}{c}
\mathbb{Z}_{X,X}(N - i)[2N - 2i] \otimes \mathbb{Z}_{W,W} \xrightarrow{\alpha_{E}^{i} \otimes \text{id}} \mathbb{Z}_{P \times S, \tilde{P}}(N)[2N] \otimes \mathbb{Z}_{W,W} \\
\downarrow \mathbb{A}_{X,W} \quad \downarrow \mathbb{A}_{P,W} \\
\mathbb{Z}_{X \times s W, \tilde{X} \times s W}(N - i)[2N - 2i] \xrightarrow{\alpha_{E}^{i} \otimes \text{id}} \mathbb{Z}_{P \times S, \tilde{P}}(N)[2N]
\end{array}
\]
This, together with the definition of \( q_* \) and \((q \times \text{id}_W)_*\), proves the result. \( \square \)

6.4. Push-forward for a projective morphism

In this section, we assume for the sake of simplicity that the base scheme \( S \) is a quasi-projective scheme over a Noetherian ring \( A \); in particular, each scheme in \( \text{Sm}_S \) admits an \( A \)-ample line bundle. Without this assumption, the results of this section remain valid for projective morphisms \( p: Y \to X \) which have a factorization as

\[ Y \xrightarrow{i} X \times_S \mathbb{P}^N_S \xrightarrow{p_1} X, \]

where \( i \) is a closed embedding.

(6.4.1)

Let \( p: Y \to X \) be a projective morphism in \( \text{Sm}_S \). By definition, there is a vector bundle \( E \to X \), with associated projective bundle \( q: \mathbb{P}(E) \to X \), and a closed embedding \( i: Y \to \mathbb{P}(E) \) such that \( p = q \circ i \). By our assumption on \( S \), there is a line bundle \( L \) on \( X \) such that \( E \otimes L \) is generated by global sections, giving the surjection

\[ \pi: 1^{N+1} \to E \otimes L \]

for some \( N \), where \( 1^{N+1} \) is the trivial rank \( N + 1 \) bundle on \( X \). This gives the closed embedding \( j: \mathbb{P}(E) \cong \mathbb{P}(E \otimes L) \to \mathbb{P}^N_X \); changing notation, we may therefore assume that \( E \) is the trivial bundle \( 1^{N+1} \) for some \( N \), giving the factorization of \( p \) as \( p = q \circ i \), with

\[ i: Y \to \mathbb{P}^N_X \]

a closed embedding, and

\[ q: \mathbb{P}^N_X \to X \]

the projection.
(6.4.2) Lemma

Suppose $X$ and $Y$ are of pure dimension $d$ and $e$ over $S$, respectively. Let $\hat{X}$ be a closed subset of $X$, $\hat{Y}$ a closed subset of $Y$ such that $p(\hat{Y}) \subset \hat{X}$. Then the composition

$$q_* \circ i_* : Z_{Y, \hat{Y}}(d - e)[2d - 2e] \to Z_{X, \hat{X}}$$

$$i: Y \to \mathbb{P}^N_X; \quad q: \mathbb{P}^N_X \to X,$$

depends only on the morphism $p$.

Proof. Suppose we have another factorization of $p$ as $q' \circ i'$, with $i': Y \to \mathbb{P}^M_X$ a closed embedding, and $q': \mathbb{P}^M_X \to X$ the projection. Let $E = 1^{N+1}$, $E' = 1^{M+1}$, so we have

$$\mathbb{P}^N_X = \mathbb{P}(E), \quad \mathbb{P}^M_X = \mathbb{P}(E'), \quad \mathbb{P}^{N+M-1}_X = \mathbb{P}(E \oplus E').$$

The projections $E \oplus E' \to E$, $E \oplus E' \to E'$ induce the closed embeddings

$$j: \mathbb{P}(E) \to \mathbb{P}(E \oplus E'); \quad j': \mathbb{P}(E') \to \mathbb{P}(E \oplus E').$$

Letting $r: \mathbb{P}(E \oplus E') \to X$ be the structure morphism, we have the commutative diagrams

$$\begin{array}{ccc}
Y & \xrightarrow{i} & \mathbb{P}(E) \\
p \downarrow & & \downarrow j \\
X & \leftarrow & \mathbb{P}(E \oplus E')
\end{array} \quad \begin{array}{ccc}
Y & \xleftarrow{i'} & \mathbb{P}(E') \\
p \downarrow & & \downarrow j' \\
X & \leftarrow & \mathbb{P}(E \oplus E')
\end{array}$$

This reduces us to considering the case of a projection

$$j: E = F \oplus F' \to F,$$

where $F$ and $F'$ are trivial bundles, giving the induced closed embedding $\tilde{j}$:

$$\begin{array}{ccc}
\mathbb{P}(F) & \xrightarrow{j} & \mathbb{P}(E) \\
q' \downarrow & & \downarrow q \\
X & = & X
\end{array}$$

and closed embedding $i': Y \to \mathbb{P}(F)$ with $i = \tilde{j} \circ i'$.

By (6.2.1) and (6.3.2), we have

$$q_* \circ i_* = q_* \circ \tilde{j}_* \circ i'_*$$

$$= q'_* \circ i'_*,$$

completing the proof. □
(6.4.3) Definition

Let \( p: Y \to X \) be a projective morphism in \( \text{Sm}_S \). Suppose \( X \) and \( Y \) are of pure dimension \( d \) and \( e \), respectively. Let \( \bar{X} \) be a closed subset of \( X \), \( \bar{Y} \) a closed subset of \( Y \) such that \( p(\bar{Y}) \subseteq \bar{X} \). Choose a trivial vector bundle \( E \to X \), with associated projective bundle \( q: \mathbb{P}(E) \to X \), and a closed embedding \( i: Y \to \mathbb{P}(E) \) such that \( p = q \circ i \). Define

\[
p_*: \mathbb{Z}_{\bar{Y}}(e - d)[2e - 2d] \to \mathbb{Z}_{\bar{X}}\bar{X}
\]

to be the composition \( q_* \circ i_* \). By (6.4.2), \( p_* \) is well-defined.

Let \( X \) be in \( \text{Sm}_S \). For integers \( N, M \geq 0 \), we let

\[
i_{N,M}: \mathbb{P}^N_X \times X \to \mathbb{P}^{N+M}_X
\]

be the Segre embedding. We let \( q^N: \mathbb{P}^N_X \to X \) denote the structure morphism, and

\[
p_1: \mathbb{P}^N_X \times X \to \mathbb{P}^N_X,
p_2: \mathbb{P}^N_X \times X \to \mathbb{P}^M_X
\]

the projections.

(6.4.4) Lemma

We have

\[
q^N_* \circ p_1* = q^M_* \circ p_2* = q^{N+M}_* \circ i_{N,M}.
\]

Proof. We give the proof without closed supports to simplify the notation.

Let \( \zeta_1 \) be the first Chern class of the tautological line bundle on \( \mathbb{P}^N_X \), \( \zeta_2 \) the first Chern class of the tautological line bundle on \( \mathbb{P}^M_X \), and \( \zeta \) the first Chern class of the tautological line bundle on \( \mathbb{P}^{N+M}_X \).

Two applications of (5.4.2) give the isomorphism

\[
\alpha^{N,M} = \sum_{i=0}^{N} \sum_{j=0}^{M} \alpha_{i,j}: \mathbb{Z}_X(N + M - i - j)[2N + 2M - 2i - 2j] \to \mathbb{Z}_{\mathbb{P}^N_X \times \mathbb{P}^M_X}(N + M)[2N + 2M],
\]

where \( \alpha_{i,j} \) is the composition

\[
\mathbb{Z}_X(N + M - i - j)[2N + 2M - 2i - 2j] \\
\xrightarrow{(q^N_* \circ p_1)^*} \mathbb{Z}_{\mathbb{P}^N_X \times \mathbb{P}^M_X}(N + M - i - j)[2N + 2M - 2i - 2j] \\
\cup (p_1^*(\zeta_1) \cup p_2^*(\zeta_2)) \xrightarrow{\cup (p_1^*(\zeta_1) \cup p_2^*(\zeta_2))} \mathbb{Z}_{\mathbb{P}^N_X \times \mathbb{P}^M_X}(N + M)[2N + 2M].
\]
It is easy to see that the composition

$$q_*^N \circ p_1* \circ \alpha^{N,M}: \bigoplus_{i=0}^N \bigoplus_{j=0}^M \mathbb{Z}_X(N + M - i - j)[2N + 2M - 2i - 2j] \to \mathbb{Z}_X$$

is projection on the factor $\mathbb{Z}_X$ ($i = N, j = M$).

Since, by (5.3.3)(iii), $\zeta_1$ and $\zeta_2$ are the classes of respective hyperplanes in $\mathbb{P}_X^N, \mathbb{P}_X^M$, we have

$$p_1^*(\zeta_1^i) \cup p_2^*(\zeta_2^j) = \text{cl}^{i+j}(L_1^i \times L_2^j),$$

where $L_1^i$ is a codimension $i$ linear subspace of $\mathbb{P}_X^N$, and $L_2^j$ is a codimension $j$ linear subspace of $\mathbb{P}_X^M$. By (6.2.3), we have

$$i_{M,N*}(p_1^*(\zeta_1^i) \cup p_2^*(\zeta_2^j)) = \text{cl}^{NM+i+j}(i_{M,N*}(L_1^i \times_L L_2^j)).$$

Applying (3.3.3), we have

$$\text{cl}^{NM+i+j}(i_{M,N*}(L_1^i \times_L L_2^j)) = \begin{cases} 2\zeta^{NM+i+j} & \text{if } i < N \text{ and } j < M \\ \zeta^{NM+i+j} & \text{if } i = N \text{ or } j = M, \end{cases}$$

so

$$i_{M,N*}(p_1^*(\zeta_1^i) \cup p_2^*(\zeta_2^j)) = \begin{cases} 2\zeta^{NM+i+j} & \text{if } i < N \text{ and } j < M \\ \zeta^{NM+i+j} & \text{if } i = N \text{ or } j = M. \end{cases} \tag{1}$$

Let $K = NM + M + N$. We have the isomorphism of (5.4.2)

$$\alpha^K = \sum_{k=0}^K \alpha_k : \bigoplus_{k=0}^K \mathbb{Z}_X(K - k)[2K - 2k] \to \mathbb{Z}_{\mathbb{P}_X^N \times \mathbb{P}_X^M}(K)[2K];$$

from the projection formula (6.2.2), and (1), we have the identity

$$i_{M,N*} \circ \alpha_{i,j} = \begin{cases} 2\alpha_{NM+i+j} & \text{if } i < N \text{ and } j < M \\ \alpha_{MN+i+j} & \text{if } i = N \text{ or } j = M. \end{cases}$$

In addition $q_*^K \circ \alpha^K$ is the projection on the factor $\mathbb{Z}_X$ (i.e., the summand $k = K$). Thus $q_*^K \circ i_{M,N*} \circ \alpha^{N,M}$ is the projection on the factor $\mathbb{Z}_X$ (i.e., the summand $i = N, j = M$), hence

$$q_*^K \circ i_{M,N*} = q_*^N \circ p_1*.$$

The identity

$$q_*^K \circ i_{M,N*} = q_*^M \circ p_2*$$

is proved similarly.
(6.4.5) Lemma

Let $E \to X$ be a vector bundle, $i: Z \to X$ a closed embedding, $i^* E \to Z$ the restriction of $E$ to $Z$. This gives the cartesian diagram

$$
\begin{array}{c}
P(i^* E) \xrightarrow{j} P(E) \\
q' \downarrow & \downarrow q \\
Z \xrightarrow{i} X
\end{array}
$$

Then

$$q_* \circ j_* = i_* \circ q'_*.$$

Proof. We give the proof without closed supports. Suppose $E$ has rank $N + 1$. We have the isomorphisms (5.4.2)

$$\alpha^E = \sum_{k=0}^{N} \alpha_k^E \oplus \oplus_{k=0}^{N} \mathbb{Z}_X(-k)[-2k] \to \mathbb{Z}_{P(E)}$$

$$\alpha^{i^* E} = \sum_{k=0}^{N} \alpha_k^{i^* E} \oplus \oplus_{k=0}^{N} \mathbb{Z}_Z(-k)[-2k] \to \mathbb{Z}_{P(i^* E)}$$

with

$$\alpha_k^E = \cup \zeta_k^E \circ q^*$$

$$\alpha_k^{i^* E} = \cup \zeta_k^{i^* E} \circ q'^*$$

By the functoriality of $c_1$ (5.3.3)(i), we have

$$i^*(\zeta_E) = \zeta_{i^* E};$$

thus, by (6.2.4), we have

$$\alpha_k^E \circ i_* = j_* \circ \alpha_k^{i^* E}.$$

This, together with the definition of $q_*$ and $q'_*$, proves the lemma. \hfill \Box

(6.4.6) Theorem

Suppose we have a sequence of projective morphisms in $\textbf{Sm}_S$

$$Z \xrightarrow{p'} Y \xrightarrow{p} X$$

with $X$, $Y$ and $Z$ of pure dimension $d$, $e$ and $f$, respectively, together with closed subsets $\hat{X}$ of $X$, $\hat{Y}$ of $Y$ and $\hat{Z}$ of $Z$, such that $p'(\hat{Z}) \subset \hat{Y}$ and $p(\hat{Y}) \subset \hat{X}$. Then the diagram

$$
\begin{array}{c}
\mathbb{Z}_{Z,\hat{Z}}(f-d)[2f-2d] \xrightarrow{p_* p'} \mathbb{Z}_{Y,\hat{Y}}(e-d)[2e-2d] \\
\downarrow (p \circ p')_* & \downarrow p_* \\
\mathbb{Z}_{X,\hat{X}} & = & \mathbb{Z}_{X,\hat{X}}
\end{array}
$$
commutes. In addition $\text{id}_{X^*} = \text{id}$.

**Proof.** The assertion $\text{id}_{X^*} = \text{id}$ follows from (6.2.1) and (6.3.6), as we may factor $\text{id}_X$ as a composition of the identity closed embedding into $X = \mathbb{P}^0_X$, followed by the identity projection $q: \mathbb{P}^0_X \to X$.

For the first assertion, choose factorizations of $p'$ and $p$ as $p = q \circ i$, $p' = q' \circ i'$, with $i: Y \to \mathbb{P}^N_X, i': Z \to \mathbb{P}^M_Y$ closed embeddings, $q: \mathbb{P}^N_X \to X, q': \mathbb{P}^M_Y \to Y$ trivial projective bundles.

We therefore arrive at the embeddings

$$j: Z \to \mathbb{P}^N_X \times_X \mathbb{P}^M_X$$
$$j': \mathbb{P}^M_Y \to \mathbb{P}^N_X \times_X \mathbb{P}^M_X$$

with

$$p \circ p' = q^N \circ p_1 \circ j$$
$$j = j' \circ i'$$
$$p_1 \circ j' = i \circ q^M.$$

This gives the factorization of $p \circ p'$ as

$$p \circ p' = q^{N+M+N+M} \circ i_{N,M} \circ j' \circ i'.$$

Thus, we have

$$(p \circ p')^* = q_{*N,M+M}^{N+M} \circ (i_{N,M} \circ j' \circ i')^*, \quad (6.4.3)$$
$$= q_{*N,M+M}^{N+M} \circ i_{N,M*} \circ j' \circ i'_*, \quad (6.2.1)$$
$$= q^N \circ p_{1*} \circ j'_* \circ i'_*, \quad (6.4.4)$$
$$= q^N \circ i_* \circ q^M \circ i'_*, \quad (6.4.5)$$
$$= p_* \circ p'_*, \quad (6.4.3).$$

\hfill \Box

(6.4.7) Theorem (Projection Formula)

Let $p: Y \to X$ be a projective morphism in $\mathcal{V}$, with $X$ and $Y$ pure dimension $d$ and $e$, respectively. Let $\hat{X}_i, i = 1, 2$, be closed subsets of $X$, $\hat{Y}_i, = 1, 2$, closed subsets of $Y$ such
that \( p(\hat{Y}_i) \subset \hat{X}_i, \ i = 1, 2 \). Then the diagram

\[
\begin{array}{ccc}
Z_{Y, \hat{Y}_1}(e - d)[2e - 2d] \otimes Z_{X, \hat{X}_2} & \xrightarrow{id \otimes p^*} & Z_{Y, \hat{Y}_1}(e - d)[2e - 2d] \otimes Z_{Y, \hat{Y}_2} \\
p_* \otimes \text{id} & & \downarrow \cup_Y \\
Z_{X, \hat{X}_1} \otimes Z_{X, \hat{X}_2} & \xrightarrow{\cup_X} & Z_{X, \hat{X}_1} \cap \hat{X}_2
\end{array}
\]

commutes.

**Proof.** This follows from the definition of \( p_* \), together with (6.2.2) and (6.3.4). \( \square \)

**Theorem (6.4.8)**

Let

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow f \\
Y & \xrightarrow{p} & X
\end{array}
\]

be a cartesian square in \( \text{Sm}_S \), with \( p \) a projective morphism. Then \( p_2 \) is a projective morphism and

\[ f^* \circ p_* = p_2^* \circ p_1^* \]

**Proof.** Let

\[
Y \xrightarrow{i} X \times_S \mathbb{P}_S^N \xrightarrow{q} X
\]

be a factorization of \( p \), with \( i \) a closed embedding and \( q \) the projection. We have the isomorphism

\[ Y \times_X Z \cong Y \times_{X \times_S \mathbb{P}_S^N} Z \times_S \mathbb{P}_S^N. \]

Let

\[ j: Y \times_X Z \rightarrow Z \times_S \mathbb{P}_S^N \]

be the map induced by the projection

\[ Y \times_{X \times_S \mathbb{P}_S^N} Z \times_S \mathbb{P}_S^N \rightarrow Z \times_S \mathbb{P}_S^N, \]

and let \( r: Z \times_S \mathbb{P}_S^N \rightarrow Z \) be the projection. We have the commutative diagram

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow f \\
Y & \xrightarrow{p} & X
\end{array}
\]

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow X \times_S \mathbb{P}_S^N \\
Y & \xrightarrow{p} & X
\end{array}
\]

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow X \times_S \mathbb{P}_S^N \\
Y & \xrightarrow{p} & X
\end{array}
\]

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow X \times_S \mathbb{P}_S^N \\
Y & \xrightarrow{p} & X
\end{array}
\]

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow X \times_S \mathbb{P}_S^N \\
Y & \xrightarrow{p} & X
\end{array}
\]

\[
\begin{array}{ccc}
Y \times_X Z & \xrightarrow{p_2} & Z \\
p_1 & \downarrow & \downarrow X \times_S \mathbb{P}_S^N \\
Y & \xrightarrow{p} & X
\end{array}
\]
with the two trapezoids cartesian; in particular, \( j \) is a closed embedding. By definition, we have

\[ p_{2*} = r_{*} \circ i_{*}; \quad p_{*} = q_{*} \circ i_{*}. \]

The result then follows from (6.2.4) and (6.3.5). \( \square \)

**THEOREM 6.4.9**

Let \( p: X \to Y \) be a projective morphism in \( \text{Sm}_S \) of relative dimension \( d \), \( \hat{X} \) a closed subset of \( X \) and \( \hat{Y} \) a closed subset of \( Y \) with \( f(\hat{X}) \subset \hat{Y} \). Let \( W \) be in \( \text{Sm}_S \), with closed subset \( \hat{W} \). Then the diagram

\[
\begin{array}{ccc}
Z_{X,\hat{X}}(d)[2d] \otimes Z_{W,\hat{W}} & \xrightarrow{p_{*} \otimes \text{id}} & Z_{Y,\hat{Y}} \otimes Z_{W,\hat{W}} \\
\downarrow & & \downarrow \\
Z_{X \times_S W,\hat{X} \times_S \hat{Y}}(d)[2d] & \xrightarrow{(p \times \text{id}_W)_{*}} & Z_{Y \times_S W,\hat{Y} \times_S \hat{W}}
\end{array}
\]

commutes.

**Proof.** This follows from (6.3.7), (6.2.7), and the definition (6.4.3) of \( p_{*} \) and \( (p \times \text{id}_W)_{*} \). \( \square \)

**THEOREM 6.4.10**

Suppose \( S = \text{Spec}(k) \), where \( k \) is a field; if \( \text{char}(k) = p > 0 \), we assume in addition that the coefficient ring \( R \) is \( \mathbb{Q} \). Let \( p: Y \to X \) be a projective morphism in \( \text{Sm}_S \) of relative dimension \( d \), \( W \) an element of \( Z^q(Y/S) \), supported on a closed subset \( \hat{Y} \) of \( Y \). Then

\[
\text{cl}^{q-d}_{p(\hat{Y})}(p_{*}(W)) = p_{*}(\text{cl}^{q}_{\hat{Y}}(W)).
\]

**Proof.** Using (6.2.3), and the functoriality of projective push-forward (6.4.6), we reduce to the case of a projective bundle \( q: \mathbb{P}_X^N \to X \). We may also assume that \( W \) is the cycle associated to an irreducible subscheme \( A \) of \( \mathbb{P}_X^N \).

Let \( B = q(A) \); we may assume that \( \hat{Y} = q^{-1}(A) \). By (6.2.5), we have

\[
H^2_B(X, R(q - d)) = 0
\]

if \( \text{codim}_X(B) > q - d \), which proves the result in this case. Suppose \( \text{codim}_X(B) = q - d \). If \( \hat{B} \) is a proper closed subset of \( B \), we have the exact sequence

\[
H^2_B(X, R(q - d)) \to H^2_B(X, R(q - d)) \to H^2_B(X \setminus \hat{B}, R(q - d))
\]

\[
\to H^2_{\hat{B}}(X, R(q - d))
\]

Applying (6.2.5), we arrive at the isomorphism

\[
H^2_B(X, R(q - d)) \to H^2_{B \setminus \hat{B}}(X \setminus \hat{B}, R(q + d));
\]
we may therefore remove from \( X \) any proper closed subset of \( B \). In particular, we may assume that \( B \) is smooth, and that \( W \) is finite over \( B \), hence \( W \) has codimension \( N \) in \( \mathbb{P}^N_B \).

We have the cartesian square

\[
\begin{array}{ccc}
\mathbb{P}^N_B & \xrightarrow{i} & \mathbb{P}^N_X \\
\downarrow{q'} & & \downarrow{q} \\
B & \xrightarrow{i} & X.
\end{array}
\]

By (6.4.8) and (6.2.5), we need only show that

\[
\text{cl}_B^0(|q'(W)|) = q'_* (\text{cl}_{\mathbb{P}^N_B}^N(|W|)).
\]

Let \( s: B \to \mathbb{P}^N_B \) be a constant section, and let \( \zeta = c_1 \mathcal{O}_{\mathbb{P}^N_B}(1) \). Since \( W \) is finite over \( B \), we have the rational equivalence

\[ W \sim_r k \cdot s(B) + W' \]

where \( W' \) is a cycle supported over a proper closed subset \( \hat{B} \) of \( B \), and \( k \) is the degree of \( W \) over \( B \). Removing \( \hat{B} \), we may assume that \( W' = 0 \). Since the cycle class respects rational equivalence, and since \( \text{cl}^N(s(B)) = \zeta^N \), we have

\[
\text{cl}_{\mathbb{P}^N_B}^N(W) = k \cdot \zeta^N \\
q'_*(W) = k \cdot B.
\]

Since \( \text{cl}_B^0(|B|) \) is the unit in \( H^*(B, R(s)) \), we have

\[
q'_*(\text{cl}_{\mathbb{P}^N_B}^N(W)) = k \cdot \text{cl}_B^0(|B|),
\]

verifying (1) and completing the proof.

(6.4.11) Definition

i) Let \( \text{Sm}_{\text{proj}} \) be the subcategory of \( \text{Sm}_S \) with the same objects, and with morphisms being the projective morphisms in \( \text{Sm}_S \) which admit a factorization as

\[
Y \xrightarrow{i} X \times_S \mathbb{P}^N_S \xrightarrow{p_1} X
\]

with \( i \) a closed embedding and \( p_1 \) the projection (if \( S \) is quasi-projective over a Noetherian ring \( A \), every projective morphism admits such a factorization).

ii) Let \( X \) be in \( \text{Sm}_S \). If \( X \) is connected, then \( X \) is equi-dimensional of dimension \( d_X \) over some connected component of \( S \); we define the Borel-Moore motive of \( X \), \( Z_X^{B,M} \), by

\[
Z_X^{B,M} = Z_X(d_X)[2d_X].
\]

We extend the definition of \( Z_X^{B,M} \) to general \( X \in \text{Sm}_S \) by taking direct sums over the connected components of \( X \).
(6.4.12) THEOREM

Sending \( X \in \text{Sm}_S \) to \( Z^B_M(X) \), and sending a projective morphism \( f:X \to Y \) to \( Z^B_M(f) \), defines a functor

\[
Z^B_M: \text{Sm}_{\text{proj}} \to D\mathcal{M}(S).
\]

Proof. This follows from (6.4.6). \( \square \)