Chapter 4
Bloch’s higher Chow groups

In this chapter, we relate the motivic cohomology groups defined in §2.2 with Bloch’s higher Chow groups.

4.1. Suspension, naive Chow groups, and the suspended cycle map
We use the technique of “relative cycles” to give a first approximation to the Chow groups of motives.

(4.1.1) Simplicial and cosimplicial objects
Let $\mathcal{C}$ be a category, $\Delta$ the category with objects the ordered sets

$[n] := \{0 < 1 < \ldots < n\}$

with morphisms order-preserving maps, $\mathcal{C}(\Delta), \mathcal{C}(\Delta^{\text{op}})$ the categories of cosimplicial, resp. simplicial objects in $\mathcal{C}$; i.e., functors

$F^*: \Delta \rightarrow \mathcal{C},$

$F_*: \Delta^{\text{op}} \rightarrow \mathcal{C},$

We have as well the full subcategory $\Delta \leq n$ with objects $[0], \ldots, [n]$, and the functor categories $\mathcal{C}(\Delta \leq n), \mathcal{C}(\Delta^{\text{op}} \leq n)$ of truncated (co)simplicial objects in $\mathcal{C}$. The inclusions

$j_n: \Delta \leq n \rightarrow \Delta$

$j_{n,m}: \Delta \leq n \rightarrow \Delta \leq m; \quad n \leq m,$

induce the restriction functors

$j_n^*: \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Delta \leq n)$

$j_{n,m}^*: \mathcal{C}(\Delta \leq m) \rightarrow \mathcal{C}(\Delta \leq n); \quad n \leq m,$

and similarly for the simplicial versions.

For an integer $n \geq 0$, we let $\Delta^{\text{n.d.}}/[n]$ be the set of injective maps

$f: [m] \rightarrow [n]$

in $\Delta$. Let

$\delta_i^m: [m] \rightarrow [m + 1]$
be the map

\[ \delta_i^m(j) = \begin{cases} 
  j & \text{if } 0 \leq j < i, \\
  j + 1 & \text{if } i \leq j \leq m.
\end{cases} \]

\textbf{(4.1.2)}

We let \( C_\mathbb{Z} \) be the free additive category generated by \( C \): objects are finite direct sums of objects of \( C \), with

\[ \text{Hom}_{C_\mathbb{Z}}(X, Y) = \mathbb{Z}[\text{Hom}_C(X, Y)] \]

for objects \( X, Y \) of \( C \), where \( \mathbb{Z}[S] \) denotes the free abelian group on a set \( S \). We may then form the category of bounded complexes \( C^b(C_\mathbb{Z}) \) and the homotopy category \( K^b(C_\mathbb{Z}) \).

These have the following universal property: Let \( F: C \rightarrow \text{Ab} \) be a functor. Then there is a unique additive extension

\[ F_\mathbb{Z}: C_\mathbb{Z} \rightarrow \text{Ab}, \]

a unique DG extension compatible with Cones:

\[ C^b(F): C^b(C_\mathbb{Z}) \rightarrow C^b(\text{Ab}), \]

and a unique exact extension

\[ K^b(F): K^b(C_\mathbb{Z}) \rightarrow K^b(\text{Ab}). \]

\textbf{(4.1.3) Complexes associated to cosimplicial objects}

Now let

\[ F_*: \Delta^{\leq n}_{\text{op}} \rightarrow C \]

be a functor. Form the object \( Z_n^\oplus(F_*) \) of \( C^b(C_\mathbb{Z}) \) by setting

\[ Z_n^\oplus(F_*)^{-m} = \bigoplus_{f:[m] \rightarrow [n]: f \in \Delta^n,d}. /[n] F_m. \]

The differential

\[ d^{-m}: Z_n^\oplus(F_*)^{-(m+1)} \rightarrow Z_n^\oplus(F_*)^{-m} \]

is given by

\[ d^{-m} = \bigoplus_{f:[m+1] \rightarrow [n], f':[m] \rightarrow [n]} d^{-m}_{f,f'}, \]

where \( d^{-m}_{f,f'} \) maps the summand \( F_{m+1} \) corresponding to \( f \) to the summand \( F_m \) corresponding to \( f' \), and

\[ d^{-m}_{f,f'} = \begin{cases} 
  (-1)^i F(\delta_i^m): F_{m+1} \rightarrow F_m & \text{if } f' = f \circ \delta_i^m \text{ for some } i, \\
  0 & \text{otherwise}.
\end{cases} \]

We have as well the object \( Z_n(F_*) \) of \( C^b(C_\mathbb{Z}) \) defined by setting

\[ Z_n(F_*)^{-m} = F_m, \]
with differential
\[ d^{-m} : \mathbb{Z}_n(F_*)^{-(m+1)} \to \mathbb{Z}_n(F_*)^{-m} \]
given by the usual alternating sum
\[ d^{-m} = \sum_{i=0}^{m} (-1)^i F(\delta_i^m) \]
 Sending \( \mathbb{Z}_n^\oplus(F_*)^{-m} \) to \( F_m \) by the sum of the projections
\[ \pi^{-m} = \sum_{f:[m] \to [n]} \pi_f: \mathbb{Z}_n^\oplus(F_*)^{-m} \to F_m \]
defines the map in \( \mathbf{C}^b(\mathbb{Z}) \)
\[ \pi_n: \mathbb{Z}_n^\oplus(F_*) \to \mathbb{Z}_n(F_*) \].
(4.1.3.1)
For \( N > n \), we let
\[ \delta_0^{N,n} : [n] \to [N] \]
be the composition
\[ \delta_0^{N-1} \circ \ldots \circ \delta_0^n. \]
(4.1.4)
Let
\[ F_* : \Delta^{\leq N \text{op}} \to \mathcal{C} \]
be a functor and take \( n < N \); one easily checks that sending \( F_m \) in the summand \( f:[m] \to [n] \) to \( F_m \) in the summand \( \delta_0^{N,n} \circ f:[m] \to [N] \) via the identity gives a map of complexes
\[ \chi^{N,n}: \mathbb{Z}_n^\oplus(j_{n,N}^*F_*) \to \mathbb{Z}_N^\oplus(F_*). \]
(4.1.4.1)
We have the canonical identification of \( \mathbb{Z}_n(j_{n,N}^*F_*) \) with the “stupid truncation” \( \sigma^\geq-n \mathbb{Z}_N(F_*) \), giving the canonical map of complexes
\[ j_{N,n*} : \mathbb{Z}_n(j_{n,N}^*F_*) \to \mathbb{Z}_N(F_*); \]
one sees immediately that the diagram
\[ \begin{array}{ccc}
\mathbb{Z}_n^\oplus(j_{n,N}^*F_*) & \xrightarrow{\chi^{N,n}} & \mathbb{Z}_N^\oplus(F_*) \\
\pi_n \downarrow & & \downarrow \pi_N \\
\mathbb{Z}_n(j_{n,N}^*F_*) & \xrightarrow{j_{N,n*}} & \mathbb{Z}_N(F_*)
\end{array} \]
commutes. Similarly, if
\[ F_* : \Delta^{\text{op}} \to \mathcal{C} \]
is a functor, we have the truncations
\[ j_N^* F_* : \Delta^{\leq N_{\text{op}}} \to C, \]
the natural map
\[ \pi_N : \mathbb{Z}_N^{\oplus}(j_N^* F_*) \to \mathbb{Z}_N(F_*). \]  
(4.1.4.2)
and the commutative diagram
\[ \begin{array}{ccc}
\mathbb{Z}_N^{\oplus}(j_N^* F_*) & \xrightarrow{\chi^{N,n}} & \mathbb{Z}_N^{\oplus}(j_N^* F_*) \\
\downarrow{\pi_n} & & \downarrow{\pi_N} \\
\mathbb{Z}_n(j_n^* F_*) & \xrightarrow{j_{N,n*}} & \mathbb{Z}_N(j_N^* F_*) 
\end{array} \]  
(4.1.4.3)
Let
\[ \sigma_i^m : [m] \to [m - 1]; \quad 0 \leq i \leq m - 1, \]  
(4.1.4.4)
be the map
\[ \sigma_i^m(j) = \begin{cases} 
  j & \text{for } 0 \leq j \leq i, \\
  j - 1 & \text{for } i < j \leq m.
\end{cases} \]
(4.1.5) Lemma
i) Let
\[ F_* : \Delta^{\leq N_{\text{op}}} \to C \]
and
\[ h : C \to \textbf{Ab} \]
be functors. Then for all \( 0 \leq n < N \), the map (see (4.1.4.1))
\[ h(\chi^{N,n}) : h(\mathbb{Z}_N^{\oplus}(j_N^* F_*)) \to h(\mathbb{Z}_N^{\oplus}(F_*)) \]
induces an isomorphism on \( H^{-m} \) for \( m < n \) and a surjection for \( m = n \).
ii) Let
\[ F_* : \Delta^{\leq n_{\text{op}}} \to C \]
and
\[ h : C \to \textbf{Ab} \]
be functors. Then the map (see (4.1.3.1))
\[ h(\pi_n) : h(\mathbb{Z}_n^{\oplus}(F_*)) \to h(\mathbb{Z}_n(F_*)) \]
induces an isomorphism on \( H^p \) for \(-n < p \leq 0\). For \( p = n \), the map
\[ H^{-n}(h(\pi_n)) : H^{-n}(h(\mathbb{Z}_n^{\oplus}(F_*))) \to H^{-n}(h(\mathbb{Z}_n(F_*))) \]
is injective, and identifies $H^{-n}(h(\mathbb{Z}_n^\oplus(F_*)))$ with

$$\cap_{i=0}^n \ker[h(F)(\delta_i^{n-1}): h(F_n) \to h(F_{n-1})].$$

**Proof.** We begin by proving (i) by induction on $N$. Using the induction hypothesis, it suffices to prove the result for $n = N - 1$, and we may assume that (i) holds for $N$ replaced by $N - 1$ and for all $n < N - 1$.

Define the complex $C^*$ by the exactness of the sequence

$$0 \to h(\mathbb{Z}_{N-1}^\oplus(F_*)) \xrightarrow{h(c^{N,N-1})} h(\mathbb{Z}_N^\oplus(F_*)) \to C^* \to 0. \quad (1)$$

For a map $f: [m] \to [m']$ in $\Delta$, we let $f + 1: [m + 1] \to [m' + 1]$ be the map

$$f + 1(j) = \begin{cases} f(j) & 0 \leq j \leq m \\ m' + 1 & j = m + 1. \end{cases}$$

Define the functor

$$G_*: \Delta^{\leq N-1 \text{op}} \to C$$

by

$$G_m = F_{m+1}$$

$$G(f: [m] \to [m']) = F(f + 1: [m + 1] \to [m' + 1]).$$

Sending an injective map $g: [m] \to [N - 1]$ to the map $g + 1: [m + 1] \to [N]$ gives an isomorphism of the quotient complex $C^*/C^0$ with $h(\mathbb{Z}_{N-1}(G_*))[-1]$.

Now suppose we have a co-cycle $\eta$ in $h(\mathbb{Z}_{N-1}^\oplus(G_*))^{-p}$, with $1 \leq p \leq N - 2$. By our induction hypothesis, applied to the functor $G_*$, there is a co-cycle $\eta_p$ in $h(\mathbb{Z}_{p}^\oplus(j_{p,N-1}(G_*)))^{-p}$ such that $\chi^{N-1,p}(\eta_p)$ and $\eta$ have the same cohomology class. Let $\tau$ be the co-cycle in $C^*$ corresponding to $\eta_p$. Then $\tau$ is an element of $h(F_{p+1})$, in the summand $\text{id}_{[p]} + 1: [p + 1] \to [N]$; the condition that $\tau$ is a co-cycle in $C^*$ is

$$h(F(\delta_i^p))(\tau) = 0; \quad i = 0, \ldots, p.$$  

Let $\rho$ be the element of $h(F_{p+2})$ defined by

$$\rho = (-1)^p h(F(\sigma_{p+1}^{p+2}))(\tau).$$

By the identities

$$\sigma_{p+1}^{p+2} \circ \delta_i^{p+1} = \delta_i^p \circ \sigma_{p}^{p+1}; \quad 0 \leq i \leq p,$$

$$\sigma_{p+1}^{p+1} \circ \delta_{p+2}^{p+1} = \text{id}_{[p+1]},$$

we have

$$h(F((-1)^i \delta_i^{p+1}))((\rho) = 0; \quad i = 0, \ldots, p,$$

$$h(F((-1)^{p+2} \delta_{p+2}^{p+1}))(\rho) = \tau,$$
Thus, if we let $\rho_N$ be the element of $C^{p+1}$ corresponding to $\rho$ in the factor
\[ \text{id}_{[p+1]} + 1: [p + 2] \to [N], \]
we have
\[ \tau = \partial_{C^*}(\rho), \]
hence $H^{-p-1}(C^*) = 0$ for $1 \leq p \leq N - 2$.

To show that $H^{-1}(C^*) = 0$, we have the exact sequence
\[ 0 \to H^{-1}(C^*) \to H^{-1}(C^*/C^0) \to C^0; \]
the same argument as above shows that $H^{-1}(C^*/C^0)$ is generated by elements $\tau \in h(F_1)$, in the summand $\text{id}_{[0]+1}: [1] \to [N]$. The condition that $\tau$ is a co-cycle in $C^*$ is just
\[ h(F(\delta_0^0))(\tau) = 0. \]

Arguing as above then shows that $\tau$ goes to zero in $H^{-1}(C^*)$. Finally, $C^0$ is the group $h(F_0)$, which we consider as being in the factor $f: [0] \to [N]$, $f(0) = N$. If $x$ is in $h(F_0)$, then, taking $y$ to be the element $h(F)(\sigma_0^0)(x)$ of $h(F_1)$, in the summand $\text{id}_{[0]+1}: [1] \to [N]$, we have as above
\[ x = \pm \partial_{C^*}(y), \]
hence $H^0(C^*) = 0$ as well.

Taking the long exact cohomology sequence associated to the sequence (1) completes the verification of the induction hypothesis, finishing the proof of (i).

For (ii), we may use (i); thus it suffices to show that the map (see (4.1.3.1))
\[ h(\pi_n): h(Z_n(F_*))^\oplus \to h(Z_n(F_*)) \]
induces an isomorphism on $H^{-p}$ for $p = n - 1$.

Let $h(Z_n(F_*))^0_0$ be the normalized co-chain complex:
\[ h(Z_n(F_*))^0_0 = \cap_{i=1}^p \ker[h(F)(\delta_i^{p-1}): h(Z_n(F_*))^{-p} \to h(Z_n(F_*))^{-p+1}] \]
with differential $\partial^{-p} = h(F)(\delta_0^{p-1})$. By the Dold-Puppe theorem, the inclusion
\[ h(Z_n(F_*))^0_0 \to h(Z_n(F_*)) \]
induces an isomorphism in cohomology $H^{-p}$ for $p \leq n - 1$. By (i), we have the exact sequence
\[ h(Z_n(F_*))^0_0 \overset{h(F)(\delta_0^{n-1})}{\to} H^{-n+1}(h(Z_n^{\oplus}(j_{n-1,n}^*(F_*)))) \]
\[ h(\chi_{n,n-1}^*): H^{-n+1}(h(Z_n^{\oplus}(F_*))) \to 0. \]
On the other hand, a direct computation shows that $h(\pi_{n-1})$ gives an identification

$$H^{-n+1}(h(\mathbb{Z}_{n-1}(j_{n-1,n}^*,F_*)))$$

$$\cong \ker[h(F)(\delta_0^{n-2}): h(\mathbb{Z}_n(F_*))_0^{-n+1} \to h(\mathbb{Z}_n(F_*))_0^{-n+2}].$$

This in turn gives, via $h(\pi_n)$, the identification of the sequence (2) with the sequence

$$h(\mathbb{Z}_n(F_*))_0^{-n} \xrightarrow{h(F)(\delta_0^{n-1})} \ker[h(F)(\delta_0^{n-2}): h(\mathbb{Z}_n(F_*))_0^{-n+1} \to h(\mathbb{Z}_n(F_*))_0^{-n+2}]$$

$$h(x_{n+1}^{n,n-1}) H^{-n+1}(h(\mathbb{Z}_n(F_*))) \to 0.$$ 

This proves that $h(\pi_n)$ gives an isomorphism on $H^{-n+1}$, and the desired injection on $H^{-n}$, completing the proof. 

\(\square\)

(4.1.6) Very smooth cosimplicial schemes

We now apply the constructions of (4.1.1)-(4.1.5) to certain cosimplicial objects in $\mathcal{V}$.

Let

$$X^* : \Delta \to \mathcal{V}$$

be a cosimplicial object in $\mathcal{V}$, such that the maps

$$X(\sigma_i^m) : X^m \to X^{m-1}$$

(cf. (4.1.4.4)) are all smooth. We call such a cosimplicial object very smooth. We now describe a lifting of $X^*$ to a cosimplicial object

$$(X^*, f_{X^*}^n) : \Delta \to \mathcal{L}(\mathcal{V}).$$ (4.1.6.1)

For each $n \geq 0$, let $X^{\leq n}$ be the disjoint union

$$X^{\leq n} = \bigsqcup_{g : [m] \to [n], g \in \Delta^{n,d}/[n]} X^m,$$

and let

$$f^n_{X^*} : X^{\leq n} \to X^n$$

be the map which is $X(g) : X^m \to X^n$ on the component indexed by $g$. This determines the object $(X^n, f^n_{X^*})$ of $\mathcal{L}(\mathcal{V})$.

For a morphism $p : [m] \to [n]$ in $\Delta$, we have the unique factorization of $p$ in $\Delta$ as

$$[m] \xrightarrow{p_{\text{sur}}} [m'] \xrightarrow{p_{\text{inj}}} [n],$$
with $p_{\text{surj}}$ surjective and $p_{\text{inj}}$ injective. Now let $h: [n] \to [n]$ be a map in $\Delta$, and let $g: [m'] \to [n']$ be an injective map in $\Delta$. We have the factorization of $(g \circ h)$ as

$$[m] \xrightarrow{(g \circ h)_{\text{surj}}} [m_{g,h}] \xrightarrow{(g \circ h)_{\text{inj}}} [n];$$

(4.1.6.2)
as each surjective map in $\Delta$ is a composition of the maps $\sigma_i^j$, the morphism

$$X((g \circ h)_{\text{surj}}): X^{m'} \to X^{m_{g,h}}$$

is a smooth morphism. Let

$$i_{g,h}: X^{m_{g,h}} \to X^{\leq n}$$

be the inclusion as the component indexed by the map $(g \circ h)_{\text{inj}}$. Let

$$q(h): X^{\leq n'} \to X^{\leq n}$$

be the map which on the component $X^{m'}$ indexed by $g: [m'] \to [n']$ is the composition $i_{g,h} \circ X((g \circ f)_{\text{surj}})$; $q(h)$ is thus a smooth morphism in $\mathcal{V}$.

The factorization (4.1.6.2) gives us the commutative diagram

$$
\begin{array}{ccc}
X^{\leq n'} & \xrightarrow{q(h)} & X^{\leq n} \\
\downarrow f_X^{n'} & & \downarrow f_X^n \\
X^{n'} & \xrightarrow{X(h)} & X^n,
\end{array}
$$

as $q(h)$ is smooth, we have the morphism

$$X(h): (X^{n'}, f_X^{n'}) \to (X^n, f_X^n)$$

in $\mathcal{V}$. Thus, sending $n$ to $(X^n, f_X^n)$, $h$ to $X(h)$, defines the desired functor (4.1.6.1).

(4.1.7) Motives associated to cosimplicial schemes

We have the functor $\mathbb{Z}(0)$:

$$\mathbb{Z}(0): \mathbb{L}(\mathcal{V})^{\text{op}} \to \mathcal{A}_{\text{mot}}(\mathcal{V})$$

$$\mathbb{Z}(0)((X, f)) = \mathbb{Z}_X(0)_f;$$

(4.1.7.1)

this extends to the functors

$$\mathbf{C}^b(\mathbb{Z}(0)): \mathbf{C}^b(\mathbb{L}(\mathcal{V})_\mathbb{Z})^{\text{op}} \to \mathbf{C}^b_{\text{mot}}(\mathcal{V})^*$$

$$\mathbf{K}^b(\mathbb{Z}(0)): \mathbf{K}^b(\mathbb{L}(\mathcal{V})_\mathbb{Z})^{\text{op}} \to \mathbf{K}^b_{\text{mot}}(\mathcal{V})^*. $$

(4.1.7.2)

If

$$X^*: \Delta \to \mathcal{V}$$
is a very smooth cosimplicial object, we have the object

\[ \mathbb{Z}^\oplus_N(j_N^*(X^*, f^*_{X^*})) \]

of \( \mathbf{C}^b(\mathcal{L}(V)_{\mathbb{Z}}) \) (cf. (4.1.1.1), (4.1.3) and (4.1.6.1)); we define \( \mathbb{Z}^\leq_N(0) \) in \( \mathbf{C}^b_{mot}(V)^* \) by

\[ \mathbb{Z}^\leq_N(0) = \mathbf{C}^b(\mathbb{Z}(0))(\mathbb{Z}^\oplus_N(j_N^*(X^*, f^*_{X^*}))) \quad (4.1.7.3) \]

Sending \( X^* \) to \( \mathbb{Z}^\leq_N(0) \) determines the functor

\[ \mathbb{Z}^\leq_N(0): \mathcal{V}(\Delta)_{\text{very smooth}} \to \mathbf{D}^b_{mot}(V)^* \quad (4.1.7.4) \]

from the category of very smooth cosimplicial schemes in \( \mathcal{V} \) to \( \mathbf{D}^b_{mot}(V)^* \); the natural maps (4.1.4.1)

\[ \chi^{N,n}: \mathbb{Z}^\oplus_n(j_n^*(X^*, f^*_{X^*})) \to \mathbb{Z}^\oplus_N(j_N^*(X^*, f^*_{X^*})) \]

give rise to the natural maps in \( \mathbf{C}^b_{mot}(V)^* \)

\[ \chi^{N,n}: \mathbb{Z}^\leq_n(0)(?) \to \mathbb{Z}^\leq_N(0)(?), \quad (4.1.7.5) \]

and define the natural transformation

\[ \chi^{N,n}: \mathbb{Z}^\leq_n(0) \to \mathbb{Z}^\leq_N(0). \quad (4.1.7.6) \]

(4.1.8) Example

Let \( \Delta^n \) be the \( S \)-scheme

\[ \Delta^n := \text{Spec}_S(\mathcal{O}_S[t_0, \ldots, t_n]/(\Sigma_i t_i - 1)). \]

We let \( v^n_i \) be the closed subscheme of \( \Delta^n \) defined by the equations

\[ t_j = 0; \quad j = 0, \ldots, n, \; j \neq i. \]

Let \( f: [m] \to [n] \) be a map in \( \Delta \). There is then a unique affine-linear map

\[ \Delta(f): \Delta^m \to \Delta^n \]

with

\[ \Delta(f)(v^m_i) = v^n_{f(i)}; \quad i = 0, \ldots, m, \]

hence we have the cosimplicial scheme

\[ \Delta^*: \Delta \to \text{Sm}_S \quad (4.1.8.1) \]
One easily sees that $\Delta^*$ gives a very smooth cosimplicial scheme in $\mathcal{V}$. We denote the maps (see (4.1.6))

$$f^*_n: \Delta^{\leq n} \to \Delta^n$$

by

$$\delta^n: \Delta^{\leq n} \to \Delta^n,$$

giving the objects (see (4.1.6.1), (4.1.7.3))

$$(\Delta^*, \delta^*): \Delta \to \mathcal{L}(\mathcal{V}),$$

$$\mathbb{Z}^N_{\Delta^*}(0) \in \mathbf{C}^b_{\text{mot}}(\mathcal{V})^*;$$

and the sequence of maps in $\mathbf{C}^b_{\text{mot}}(\mathcal{V})^*$ (see (4.1.7.5))

$$\ldots \to \mathbb{Z}^N_{\Delta^*}(0) \to \mathbb{Z}^{N+1}_{\Delta^*}(0) \to \mathbb{Z}^{N+2, N+1}_{\Delta^*} \to \ldots$$

(4.1.9) Definition

i) Let $\Gamma$ be an object of $\mathbf{C}^b_{\text{mot}}(\mathcal{V})^*$. We let $\Sigma^N \Gamma$ denote the object (see (4.1.8.2))

$$\mathbb{Z}^N_{\Delta^*}(0) \times \Gamma[-N].$$

where $\times$ is the tensor operation in $\mathbf{C}^b_{\text{mot}}(\mathcal{V})^*$. Sending $\Gamma$ to $\Sigma^N \Gamma$ gives the functor

$$\Sigma^N: \mathbf{C}^b_{\text{mot}}(\mathcal{V})^* \to \mathbf{C}^b_{\text{mot}}(\mathcal{V})^*;$$

we have as well the extension of $\Sigma^N$ to exact functors

$$\Sigma^N: \mathbf{K}^b_{\text{mot}}(\mathcal{V})^* \to \mathbf{K}^b_{\text{mot}}(\mathcal{V})^*;$$

and

$$\Sigma^N: \mathbf{D}^b_{\text{mot}}(\mathcal{V})^* \to \mathbf{D}^b_{\text{mot}}(\mathcal{V})^*.$$

ii) Let

$$\Gamma_*: \Delta^{\text{op}} \to \mathbf{C}^b_{\text{mot}}(\mathcal{V})^*$$

be a simplicial object of $\mathbf{C}^b_{\text{mot}}(\mathcal{V})^*$. Let $\mathcal{Z}_{\text{mot}}(\Gamma_*)$ be the total complex of the double complex

$$\ldots \to \mathcal{Z}_{\text{mot}}(\Gamma_n)[n] \xrightarrow{d_{-n}} \mathcal{Z}_{\text{mot}}(\Gamma_{n-1})[n-1] \to \ldots \xrightarrow{d_{-1}} \mathcal{Z}_{\text{mot}}(\Gamma_0),$$

where $d_{-n}$ is the usual alternating sum

$$d_{-n} = \sum_{i=0}^{n} (-1)^i \mathcal{Z}_{\text{mot}}(\Gamma(\delta_{i}^{n-1})).$$
iii) We let $Z_{\Delta^*}(0)_{\delta^*}$ denote the simplicial object $Z(0)((\Delta^*, \delta^*))$:

$$Z_{\Delta^*}(0)_{\delta^*} : \Delta^{op} \to A_{mot}(\mathcal{V})^*.$$

The sequence of maps (4.1.8.3) gives the sequences of natural transformations

$$\ldots \chi^{N,N-1} \to \Sigma^N [N] \chi^{N+1,N} \to \Sigma^{N+1}[N+1] \chi^{N+2,N+1} \to \ldots$$  \hspace{1cm} (4.1.9.1)

We let

$$i_N : \text{id} \to \Sigma^N [N]$$  \hspace{1cm} (4.1.9.2)

be the composition

$$i_N := \chi^{N,N-1} \circ \ldots \circ \chi^{1,0}.$$

If $\Gamma_*$ is a simplicial object of $A_{mot}(\mathcal{V})^*$, and $X$ is an object of $C_{mot}^b(\mathcal{V})^*$, we have the simplicial object $X \times \Gamma_*$ of $C_{mot}^b(\mathcal{V})^*$.

(4.1.10) DEFINITION

Let $X$ be an object of $C_{mot}^b(\mathcal{V})^*$. Define the complex $Z_{mot}(X, *)$ to be the total complex associated to the simplicial object

$$Z_{mot}(X \times Z_{\Delta^*}(0)_{\delta^*})$$

of $C^b(\text{Ab})$. (see (4.1.9)(ii),(iii)).

Sending $X$ to $Z_{mot}(X, *)$ defines the DG functor

$$Z_{mot}(\cdot) : C_{mot}^b(\mathcal{V})^* \to C^-(\text{Ab})$$  \hspace{1cm} (4.1.10.1)

and extends to the exact functor

$$Z_{mot}(\cdot) : K_{mot}^b(\mathcal{V})^* \to K^-(\text{Ab})$$  \hspace{1cm} (4.1.10.2)

The natural maps (4.1.4.2) give the natural maps

$$\pi_N \times \text{id}_{\Gamma} : Z_{\Delta^*}^{\leq N}(0) \times \Gamma \to Z_N(Z_{\Delta^*}(0)_{\delta^*}) \times \Gamma$$  \hspace{1cm} (4.1.10.3)

which in turn give the natural transformation

$$\Pi_N : Z_{mot} \circ \Sigma^N (-)[-N] \to Z_{mot}(-, \cdot)$$  \hspace{1cm} (4.1.10.4)

by applying the functor $Z_{mot}$ to the natural maps $\pi_N \times \text{id}_{\Gamma}$, and composing with the natural inclusion

$$Z_{mot}(\Sigma_N(Z_{\Delta^*}(0)_{\delta^*}) \times \Gamma) = \sigma^{\geq -N} Z_{mot}(\Gamma, \cdot) \subset Z_{mot}(\Gamma, \cdot).$$
The commutativity of the diagram (4.1.4.3) gives the relation
\[ \Pi_{N+1} \circ \mathcal{Z}_{\text{mot}}(\chi^{N+1,N}) = \Pi_N. \] (4.1.10.5)

**Proposition**

Suppose \( S = \text{Spec}(k) \), and \( X \) is a smooth variety over \( k \). Then \( \mathcal{Z}_{\text{mot}}(\mathbb{Z}_X(q)[2q], \ast) \) is naturally isomorphic to Bloch’s cycle complex \( z^q(X, \ast) \) (see [B]).

**Proof.** We have the identity
\[ \mathbb{Z}_X(q)f[2q] \times \mathbb{Z}_{\Delta^p}(0)_{\delta^p} = \mathbb{Z}_{X \times_k \Delta^p}(q)_{\text{id}_X \times \delta^p}[2q], \]
giving the identification of \( \mathcal{Z}_{\text{mot}}(\mathbb{Z}_X(q)[2q], -p) \) with the subgroup \( \mathbb{Z}^q(X \times_k \Delta^p)_{\delta^p} \) of \( \mathbb{Z}^q(X \times_k \Delta^p/k) \) generated by effective cycles \( W \) such that \((\text{id}_X \times f)^*(W)\) is defined for all face maps
\[ f: \Delta^m \to \Delta^p. \]

This is the same as the group \( z^q(X, p) \) defined in [B]. With the shift \([2q]\), the graded group \( \mathcal{Z}_{\text{mot}}(\mathbb{Z}_X(q)[2q], -p) \) is concentrated in degree \(-p\). The coboundary map
\[ d^{-p}: \mathcal{Z}_{\text{mot}}(\mathbb{Z}_X(q)[2q], -p) \to \mathcal{Z}_{\text{mot}}(\mathbb{Z}_X(q)[2q], -p + 1) \]
are given as the alternating sum of the restrictions to the codimension one faces of \( X \times_k \Delta^p \), which is the same as the boundary map
\[ d_p: z^q(X, p) \to z^q(X, p + 1), \]
completing the proof. \( \square \)

**Lemma**

i) For each \( \Gamma \) in \( \mathcal{C}_{\text{mot}}(V)^* \), there is an integer \( N_\Gamma \) such that the maps
\[
\begin{align*}
H^0(\Pi_N(\Gamma)) & : H^0(\mathcal{Z}_{\text{mot}}(\Sigma^N(\Gamma)[N])) \to H^0(\mathcal{Z}_{\text{mot}}(\Gamma, \ast)) \\
H^0(\chi^{N+1,N}) & : H^0(\mathcal{Z}_{\text{mot}}(\Sigma^N(\Gamma)[N])) \to H^0(\mathcal{Z}_{\text{mot}}(\Sigma^{N+1}(\Gamma)[N + 1]))
\end{align*}
\]
are isomorphisms for all \( N \geq N_\Gamma \). In addition, if we take \( N_\Gamma \) minimal, we have
\[ N_{\Gamma[-1]} = N_\Gamma + 1. \]

ii) For each pair of integers \((p, q)\), there is an integer \( N_{p,q} \) such that the maps
\[
\begin{align*}
H^0(\Pi_N(\Gamma)) & : H^0(\mathcal{Z}_{\text{mot}}(\Sigma^N(\Gamma)[N])) \to H^0(\mathcal{Z}_{\text{mot}}(\Gamma, \ast)) \\
H^0(\chi^{N+1,N}) & : H^0(\mathcal{Z}_{\text{mot}}(\Sigma^N(\Gamma)[N])) \to H^0(\mathcal{Z}_{\text{mot}}(\Sigma^{N+1}(\Gamma)[N + 1]))
\end{align*}
\]
are isomorphisms for all \( N \geq N_{p, q} \), and for all \( \Gamma \) of the form
\[
\Gamma = \mathbb{Z}_X(q)_f[m]
\]
with \( m \geq p \).

**Proof.** The assertion (i) for \( \Gamma \) in \( A_{mot}(V)^* \) follow from (4.1.5), with
\[
\mathcal{C} = A_{mot}(V)^*, \ X = \Gamma \times j_{N+2}^* \mathbb{Z}_{\Delta^*}(0), \ h = \mathbb{Z}_{mot};
\]
we may take \( N_p = 2q - p + 1 \) for \( \Gamma = \mathbb{Z}_X(q)_f[p] \) in \( A_{mot}(V)^* \). Thus, taking \( N_{p, q} = 2q - p + 1 \) proves (ii).

As the extension of both functors from \( A_{mot}(V)^* \) to \( C_{b mot}(V)^* \) preserves the operation of taking Cones, and as \( C_{b mot}(V)^* \) is generated by \( A_{mot}(V)^* \) via the operation of taking Cones, the assertion (i) is also true for arbitrary \( \Gamma \) in \( C_{b mot}(V)^* \).

(4.1.13) **Definition**

Let \( \Gamma \) be an object of \( C_{b mot}(V)^* \). The **naive higher Chow groups of \( \Gamma \),** \( CH_{naive}(\Gamma, p) \), are defined by
\[
CH_{naive}(\Gamma, p) = H^{-p}(\mathbb{Z}_{mot}(\Gamma, *))
\]
We often write \( CH_{naive}(\Gamma) \) for \( CH_{naive}(\Gamma, 0) \)

It follows from (4.1.11) that there is a natural isomorphism
\[
CH_{naive}(\mathbb{Z}_X(q)[2q], p) \cong CH^q(X, p) \tag{4.1.13.1}
\]
for \( X \) a smooth \( k \)-variety, in case \( S = \text{Spec}(k), k \) a field, where the group \( CH^q(X, p) \) is **Bloch’s higher Chow group** (see [B])
\[
CH^q(X, p) := H_p(z^q(X, *)).
\]

From (4.1.12), we have a natural isomorphism
\[
CH_{naive}(\Gamma, p) \cong H^{-p}(\mathbb{Z}_{mot}(\Sigma^N(\Gamma)[N])) \tag{4.1.13.2}
\]
for all \( N \geq N_p + p \).

Sending \( \Gamma \) to \( CH_{naive}(\Gamma) \) defines a cohomological functor
\[
CH_{naive}(-): K_{b mot}(V)^* \to \text{Ab}.
\]

The sequence of natural transformations (4.1.9.1) defines the cohomological functor (for each \( a \geq 0 \))
\[
\lim_{\overset{\longrightarrow}{N}} \text{Hom}(e^{\otimes a} \otimes 1, \Sigma^N(-)) : K_{b mot}(V)^* \to \text{Ab},
\]
\[
\Gamma \mapsto \lim_{\overset{\longrightarrow}{N}} \text{Hom}_{K_{b mot}(V)}(e^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \tag{4.1.13.3}
\]
Proposition

i) There is a natural exact isomorphism of cohomological functors from $K^b_{mot}(\mathcal{V})^*$ to $\text{Ab}$:

$$\Sigma^*[\ast] \text{cyc}: \text{CH}_{\text{naive}}(-) \to \lim_{N,a} \text{Hom}(e^{\otimes a} \otimes 1, \Sigma^N[N](-)),$$

The limit on the right is constant after a finite stage for each object $\Gamma$ of $K^b_{mot}(\mathcal{V})^*$.

Proof. It follows from (3.2.4) that the functor $\mathcal{Z}_{mot}$ gives an isomorphism

$$\text{Hom}_{K^b_{mot}(\mathcal{V})}(e^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \to H^0(\mathcal{Z}_{mot}(\Sigma^N(\Gamma)[N]));$$

for all $a$ sufficiently large. By (4.1.12), this, combined with the natural transformation of that lemma, gives the natural isomorphism

$$\lim_{N,a} \text{Hom}(e^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \to \text{CH}_{\text{naive}}(\Gamma);$$

we take $\Sigma^*[\ast] \text{cyc}(\Gamma)$ to be the inverse of this isomorphism. \qed

Lemma

The sequence of natural transformations (4.1.9.1) composed with the functor

$$K^b_{mot}(\mathcal{V})^* \to D^b_{mot}(\mathcal{V})^*$$

is a sequence of natural isomorphisms.

Proof. Let $X$ be an object in an additive category $\mathcal{A}$. We have the constant functors

$$X^{\leq N}; \Delta^{\leq N} \to \mathcal{A}$$

We have the natural map (4.1.4.1)

$$\chi^{N+1,N}_X: Z^+_N(X^{\leq N}) \to Z^+_N(X^{\leq N+1});$$

let $B^{N+1}(X)$ denote the cone

$$B^{N+1}(X) = \text{Cone}(\chi^{N+1,N}_X);$$

we set $B^0(X) = X$. In $K^b(\mathcal{A})$, we have the natural isomorphism

$$B^{N+1}(X) \cong \text{Cone}(\text{id}: B^N(X) \to B^N(X))[-1]$$

showing that $B^N(X)$ is isomorphic to zero for all $N \geq 1$. This implies that the map $\chi^{N+1,N}_X$ is an isomorphism in $K^b(\mathcal{A})$ for all $N \geq 0$. 

The homotopy axiom (2.1.3)(a), combined with the moving lemma axiom (2.1.3)(e), shows that the map
\[ p^*_n: \mathbb{Z}_S(0) \to \mathbb{Z}_{\Delta^n}(0) \]
is an isomorphism for each \( n \) in \( D^b_{mot}(\mathcal{V})^* \). Thus we have the isomorphism
\[ p^*_n: \mathbb{Z}^\otimes_N \mathbb{Z}_S(0) \leq N \to \mathbb{Z}^\leq N_{\Delta^n}(0) \]
in \( D^b_{mot}(\mathcal{V})^* \). The remarks of the previous paragraph then show that the map
\[ \chi^{N+1}: \mathbb{Z}^\leq N_{\Delta^n}(0) \to \mathbb{Z}^\leq N_{\Delta^{n+1}}(0) \]
is an isomorphism in \( D^b_{mot}(\mathcal{V})^* \), as claimed. \( \square \)

We have the map (4.1.9.2)
\[ i_N(\Gamma): \Gamma \to \Sigma^N(\Gamma)[N]; \]
by (4.1.15) \( i_N(\Gamma) \) is an isomorphism in \( D^b_{mot}(\mathcal{V}) \). We define the naive cycle class map
\[ \text{cl}_{naive}(\Gamma): CH_{naive}(\Gamma) \to \text{Hom}_{D^b_{mot}(\mathcal{V})}(1, \Gamma) \]  
(4.1.15.1)
as the composition
\[
CH_{naive}(\Gamma) \xrightarrow{\Sigma^*[\ast]cyc(\Gamma)} \lim_{N,a} \text{Hom}(\varepsilon^a \otimes 1, \Sigma^N(\Gamma)[N]) \\
= \text{Hom}_{K^b_{mot}(\mathcal{V})}(\varepsilon^a \otimes 1, \Sigma^N(\Gamma)[N]) \to \text{Hom}_{D^b_{mot}(\mathcal{V})}(\varepsilon^a \otimes 1, \Sigma^N(\Gamma)[N]) \\
\xrightarrow{\nu_{a,0}^{-1} \circ (-)\circ i_N^0(\Gamma)^{-1}} \text{Hom}_{D^b_{mot}(\mathcal{V})}(1, \Gamma).
\]
where \( N \) is any integer \( \geq N\Gamma, \) \( a \) is sufficiently large (depending only on \( \Gamma \)) and \( \nu_a \) is the isomorphism (2.2.4.1).

(4.1.16) Remark

i) Let \( X \) be in \( \mathcal{V} \). The map
\[ \delta^1: \Delta^\leq 1 \to \Delta^1 \]
is the union \( \text{id}_{\Delta^1} \cup i_1 \cup i_0 \), where \( i_0: S \to \Delta^1, i_1: S \to \Delta^1 \) are the sections with value \( v_0 \).
and $v_1$. We have the commutative diagram with exact columns

$$
\begin{array}{ccc}
\mathbb{Z}_{mot}(\mathbb{Z}_X(q)[2q] \times \mathbb{Z}_{\Delta^1}(0)_{\delta^1}) & \xrightarrow{d^{-1}} & \mathbb{Z}^q(X \times \Delta^1)_{\id_X \times \delta^1} \\
\downarrow & & \downarrow i_i^* - i_0^* \\
\mathbb{Z}_{mot}(\mathbb{Z}_X(q)[2q]) & \xrightarrow{\Delta} & \mathbb{Z}^q(X/S) \\
\downarrow & & \downarrow \\
\text{CH}_{naive}(\mathbb{Z}_X(q)[2q]) & = & \text{CH}_{naive}(\mathbb{Z}_X(q)[2q]) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}
$$

If $S = \text{Spec}(k)$, we have, via (4.1.11), the identification of $\text{CH}_{naive}(\mathbb{Z}_X(q)[2q])$ with the classical Chow group $\text{CH}^q(X)$, and the bottom row in (1) is the standard sequence defining $\text{CH}^q(X)$. We may take this as the definition of the naive Chow group $\text{CH}^q_{naive}(X/S)$ for an arbitrary base $S$.

ii) The cycle map $\Sigma^*[\cdot] \text{cyc}(\Gamma)$ (4.1.14) and the cycle map $\text{cyc}_\Gamma$ (3.3.1.3) are compatible in the following way: We have the commutative diagram

$$
\begin{array}{ccc}
H^0(\mathbb{Z}_{mot}(\Gamma)) & \xrightarrow{\text{cyc}_\Gamma} & \text{Hom}_{K_{mot}^b}(\nu)(\mathcal{E}^{\otimes a} \otimes 1, \Gamma) \\
\chi^{N,0} \downarrow & & \downarrow \chi^{N,0} \\
H^0(\mathbb{Z}_{mot}(\Sigma^N \Gamma[N])) & \xrightarrow{\text{cyc}_{\Sigma^N \Gamma[N]}} & \text{Hom}_{K_{mot}^b}(\nu)(\mathcal{E}^{\otimes a} \otimes 1, \Sigma^N \Gamma[N]).
\end{array}
$$

For $N \geq N\Gamma$, we have the isomorphism

$$
\Pi_N: H^0(\mathbb{Z}_{mot}(\Sigma^N \Gamma[N])) \to \text{CH}_{naive}(\Gamma)
$$

and

$$
\Sigma^*[\cdot] \text{cyc}(\Gamma) = \text{cyc}_{\Sigma^N \Gamma[N]} \circ (\Pi_N)^{-1}.
$$

For $\Gamma = \mathbb{Z}_X(q)[2q]$, this gives us the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}^q(X/S) = \mathbb{Z}_{mot}(\mathbb{Z}_X(q)[2q]) & \xrightarrow{\text{cyc}_X} & \text{Hom}_{K_{mot}^b}(\nu)(\mathcal{E}^{\otimes a} \otimes 1, \mathbb{Z}_X(q)[2q]) \\
\downarrow & & \downarrow \chi^{1,0} \\
\text{CH}_{naive}^q(X/S) & \xrightarrow{\Sigma^*[\cdot] \text{cyc}(\mathbb{Z}_X(q)[2q])} & \text{Hom}_{K_{mot}^b}(\nu)(\mathcal{E}^{\otimes a} \otimes 1, \Sigma^1 \mathbb{Z}_X(q)[2q + 1]),
\end{array}
$$

where the left-hand vertical arrows is the surjection of (i).

iii) We define the naive higher Chow groups of $X$, for $X \in \mathcal{V}$, as

$$
\text{CH}_{naive}^q(X/S, p) = \text{CH}_{naive}(\mathbb{Z}_X(q)[2q - p]).
$$

By the isomorphism (4.1.13.1), this agrees with Bloch’s higher Chow groups in case $S = \text{Spec}(k)$. 

[Q.E.D.]
4.2. Hyper-resolutions and the Chow group

For a base scheme $S$ of the form $\text{Spec}(k)$, the localization theorem for the higher Chow groups (see [B2]) shows that the naive Chow groups $\text{CH}^r_{\text{naive}}(X, p)$ may be also defined as the hypercohomology on $X$ of the complex of sheaves associated to the presheaf

$$U \mapsto z^q(U, \ast)$$

For a general base scheme $S$, the analogous statement is probably not true for arbitrary $X$; we must therefore pass from the complex $Z_{\text{mot}}(\mathbb{Z}_X(q)_{F}[2q], \ast)$ to the associated complex of sheaves on $X$ and take hypercohomology to get the proper definition of the higher Chow groups of $X$ over $S$. In order to have a reasonable understanding of this operation and how it affects the maps in the category $D^b_{\text{mot}}(\mathcal{V})$, we need to define the notions of associated sheaf and hypercohomology for the complexes $Z_{\text{mot}}(\Gamma, \ast)$ for arbitrary objects $\Gamma$ of $C_{\text{mot}}^b(\mathcal{V})^*$. This is accomplished via the notions of a Čech resolution, and a hyper-resolution, of $\Gamma$.

(4.2.1) Motives of non-degenerate simplicial schemes

Let $\Delta_{n.d.}$ be the subcategory of injective maps in $\Delta$. We call a functor

$$X_*: \Delta_{n.d.}^{\text{op}} \to \mathcal{C}$$

$$X^*: \Delta_{n.d.} \to \mathcal{C}$$

a non-degenerate simplicial object (resp. non-degenerate cosimplicial object) of $\mathcal{C}$. Let

$$(X_*, f_*): \Delta_{n.d.}^{\text{op}} \to \mathcal{L}(\mathcal{V})$$

be a non-degenerate simplicial object of $\mathcal{L}(\mathcal{V})$, with

$$(X_*, f_*)_m = (X_m, f_m)$$

for $m = 0, 1, \ldots$. This gives us the non-degenerate cosimplicial object

$$Z_{X_*}(q)_{f_*}: \Delta_{n.d.} \to A_{\text{mot}}(\mathcal{V})^*$$

of $A_{\text{mot}}(\mathcal{V})^*$ with

$$Z_{X_*}(q)_{f_*}^m = Z_{X_m}(q)_{f_m}.$$  

For each $N \geq 0$, we may then form the truncated complex

$$Z_{X_*}(q)_{f_*}^{\ast \leq N} \in C_{\text{mot}}^b(\mathcal{V})^*$$

which are the object $Z_{X_*}(q)_{f_*}^m$ in degree $m$, and with co-boundary the usual alternating sum.
(4.2.2) Čech resolutions

Let $(X, f)$ be in $\mathcal{L}(\mathcal{V})$, and $\mathcal{U} := \{U_0, \ldots, U_m\}$ a Zariski open cover of $X$. For an ordered index $I = (i_0 < \ldots < i_k)$, with $0 \leq i_j \leq m$, we let $U_I$ denote the intersection

$$U_I = U_{i_0} \cap \ldots \cap U_{i_m}.$$ 

We have the associated augmented simplicial scheme

$$j_{\mathcal{U}}: \mathcal{U}_* \to X$$

where $\mathcal{U}_*$ is the simplicial scheme with non-degenerate $k$-simplices

$$\mathcal{U}_*^{n.d.} = \coprod_{I = (i_0 < \ldots < i_k)} U_I;$$

let

$$j_m: \mathcal{U}_m^{n.d.} \to X$$

be the union of the inclusions. This gives us the non-degenerated simplicial scheme $\mathcal{U}_*^{n.d.}$ (which is the empty scheme for $n > m$). We may then lift $\mathcal{U}_*^{n.d.}$ to the non-degenerate simplicial object

$$(\mathcal{U}_*, j^* f)^{n.d.}: \Delta^{op}_{n.d.} \to \mathcal{L}(\mathcal{V})$$

with $j^* f_m = j_m^* f$. We let

$$\mathbb{Z}_{\Delta^{op}_{n.d.} (q)_f^{* \leq n}} \quad (4.2.2.1)$$

be the corresponding object of $\mathbf{C}_m^{b, mot}(\mathcal{V})$; clearly the objects $\mathbb{Z}_{\Delta^{op}_{n.d.} (q)_f^{* \leq n}}$ are all canonically isomorphic for $n \geq m$. For $n \geq m$, we denote the complex (4.2.2.1) by $\mathbb{Z}_{\Delta^{op}_{n.d.} (q)_f}$.

The Mayer-Vietoris sequence (2.2.6.1) shows that the augmentation induces an isomorphism

$$j_{X, \mathcal{U}}^*: \mathfrak{e}^{\otimes a} \otimes \mathbb{Z}_X(q)_f[p] \to \mathfrak{c}^{\otimes a} \otimes \mathbb{Z}_{\mathcal{U}_m^{n.d.} (q)_f}[p]. \quad (4.2.2.2)$$

in $\mathbf{D}^{b, mot}(\mathcal{V})$ for all $a \geq 0$.

We call the map (4.2.2.2) in $\mathbf{C}_m^{b, mot}(\mathcal{V})$ a Čech resolution of $\mathfrak{c}^{\otimes a} \otimes \mathbb{Z}_X(q)_f[p]$; we extend the notion of a Čech resolution to arbitrary objects of $\mathcal{A}_m^{mot}(\mathcal{V})$ by taking direct sums.

If we have a Čech resolution

$$j_{X, \mathcal{U}}^*: \mathfrak{e}^{\otimes a} \otimes \mathbb{Z}_X(q)_f[p] \to \mathfrak{c}^{\otimes a} \otimes \mathbb{Z}_{\mathcal{U}_m^{n.d.} (q)_f}[p].$$

coming from a cover $\mathcal{U}$ of $X$, each refinement

$$\rho: \mathcal{V} \to \mathcal{U}$$


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gives rise to a commutative diagram

\[ \mathfrak{e}^\otimes a \otimes Z_X(q)_f[p] \xrightarrow{j_{X,U}} \mathfrak{e}^\otimes a \otimes ZU^*_f(q)_f[p] \]

\[ \mathfrak{e}^\otimes a \otimes Z_X(q)_f[p] \xrightarrow{j_{X,U}} \mathfrak{e}^\otimes a \otimes ZU^*_f(q)_f[p] \]

\[ \vdash \]

\[ \mathfrak{e}^\otimes a \otimes Z_X(q)_f[p] \xrightarrow{j_{X,U}} \mathfrak{e}^\otimes a \otimes ZU^*_f(q)_f[p] \]

\[ \mathfrak{e}^\otimes a \otimes Z_X(q)_f[p] \xrightarrow{j_{X,U}} \mathfrak{e}^\otimes a \otimes ZU^*_f(q)_f[p] \]

(4.2.3) Definition

Let \( \Gamma = \bigoplus_{i=1}^m e^\otimes a_i \otimes Z_{X_i}(q_i)[p_i]_f \) be an object of \( A_{mot}(\mathcal{V})^* \). A Zariski open cover of \( \Gamma \) consists of a finite Zariski open cover \( \mathcal{U}_i = \{U_{0i}, \ldots, U_{ni}\} \) of \( X_i \) for each \( i = 1, \ldots, m \). If \( \mathcal{U} \) is a Zariski open cover of \( \Gamma \), a refinement \( \rho: \mathcal{W} \rightarrow \mathcal{U} \) is a collection of refinements \( \rho_i: \mathcal{W}_i \rightarrow \mathcal{U}_i \) for each \( i = 1, \ldots, m \).

If \( \mathcal{U} \) is a Zariski open cover of \( \Gamma \in A_{mot}(\mathcal{V})^* \), the direct sum of the Čech resolutions for each component \( e^\otimes a_i \otimes Z_{X_i}(q_i)[p_i]_f \) gives a Čech resolution of \( \Gamma \). We denote this map in \( C_{mot}(\mathcal{V})^* \) by

\[ j_{\mathcal{U}, \Gamma}: \Gamma \rightarrow \mathcal{U} \]

(4.2.4) A structural result

Before proceeding further, we need to examine the morphisms in the category \( A_{mot}(\mathcal{V})^* \) a bit more closely. We recall that the category \( A_{mot}(\mathcal{V}) \) is constructed from the tensor category \( A_2(\mathcal{V}) \) by taking the co-product with the DG tensor category \( E \), and then adjoining morphisms (see (1.2.5), (1.2.7) and (1.2.8)). By (II, (2.4.11)(i) and (ii)) there is a graded symmetric semi-monoidal category \( C \), with objects generated by a single object \( e \), such that, as a graded tensor category without unit, we have

\[ E = C \]

i.e., \( E \) is the graded additive category generated by \( C \), and the tensor structure on \( E \) is induced by the symmetric monoidal structure of \( C \).

As a graded symmetric semi-monoidal category, there the a natural map (see II, (2.4.4))

\[ \pm 1 \times S_n \rightarrow \text{Hom}_C(e^\otimes n, e^\otimes n) \]

which, when extended to \( E \), is the restriction of the canonical map

\[ \mathbb{Z}[S_n] \rightarrow \text{Hom}_E(e^\otimes n, e^\otimes n) \]

sending \( \sigma \in S_n \) to the symmetry isomorphism \( \tau_\sigma \). In addition to the structure results described in the previous paragraph, a set of representatives in

\[ \text{Hom}_C(e^\otimes n, e^\otimes n)^q \]

for the action of \( \pm 1 \times S_n \) (by either left or right composition) forms a \( \mathbb{Z}[S_n] \)-basis of \( \text{Hom}_E(e^\otimes n, e^\otimes n)^q \) (cf. (II, (2.4.9)(iii))).
Now suppose we have connected schemes $X$ and $Y$ in $\mathcal{V}$, and a morphism

$$q: \epsilon^{\otimes a} \otimes \mathbb{Z}_Y (b)[c] \rightarrow \epsilon^{\otimes a'} \otimes \mathbb{Z}_X (b')[c']$$

in $A_{\text{mot}}(\mathcal{V})^*$. By (II, (1.7.2)), we may write each such map $q$ uniquely as a sum with $\mathbb{Z}$-coefficients of compositions

$$\epsilon^{\otimes a} \otimes \mathbb{Z}_Y (b)[c]\tau\text{id}_Y \rightarrow \epsilon^{\otimes a'} \otimes \mathbb{Z}_Y (b)[c + r] = \epsilon^{\otimes a'} \otimes \epsilon^{\otimes a-a'} \otimes \mathbb{Z}_Y (b)[c + r]$$

$$\text{id}_{\epsilon^{\otimes a'}} \otimes h_1 \otimes \cdots \otimes h_s \text{id}_{\epsilon^{\otimes a'}} \otimes \mathbb{Z}_{W_1} (b_1, g_1, c_1) \otimes \cdots \otimes \mathbb{Z}_{W_s} (b_s, g_s, c_s) \otimes \mathbb{Z}_Y (b)[c + r]$$

$$\text{id}_{\epsilon^{\otimes a'}} \otimes \mathbb{Z}_{W_1} \times \cdots \times W_s \times Y \rightarrow \epsilon^{\otimes a'} \otimes \mathbb{Z}_{W_1} \times \cdots \times W_s \times Y (\sum_i b_i + b) \times g_i \times g_s \times g [\sum_i c_i + c + r]$$

$$\text{id}_{\epsilon^{\otimes a'}} \otimes \mathbb{Z}_X (b')[c'].$$

(4.2.4.1)

Here the $h_i$ are maps

$$h_i: \epsilon^{\otimes e_i} \rightarrow \mathbb{Z}_{W_i} (b_i, g_i, c_i)$$

adjoined in (1.2.5), (1.2.7) and (1.2.8), we have

$$\sum_{i=1}^s e_i = a = a'; \quad b_1 + \cdots + b_s + b = b'; \quad \sum_i c_i + c + r = c'$$

and

$$p: (X, f) \rightarrow (W_1 \times S \times \cdots \times W_s \times S \times Y, g_1 \times \cdots \times g_s \times g)$$

is a map in $\mathcal{L}(\mathcal{V})$. The map

$$\tau: \epsilon^{\otimes a} \rightarrow \epsilon^{\otimes a}[p]$$

is a map in $\mathcal{C}$.

We may form a $\mathbb{Z}$-basis of

$$\text{Hom}_{A_{\text{mot}}(\mathcal{V})^*} (\epsilon^{\otimes a} \otimes \mathbb{Z}_Y (b)[c], \epsilon^{\otimes a'} \otimes \mathbb{Z}_X (b')[c'])$$

consisting of compositions of the form (4.2.4.1) by ordering the set of adjoined maps $h$, taking $h_1 \leq \ldots \leq h_s$, and taking $\tau$ in a chosen set of representative of $\text{Hom}_\mathcal{C}(\epsilon^{\otimes n}, \epsilon^{\otimes n})^* \setminus \ast$ modulo the action of the $\pm 1 \times S(h_*)$, where

$$S(h_*) \subset S_n$$

is the group of order-preserving permutations of $\{h_1, \ldots, h_s\}$ (see (II, loc. cit.)) for details.

For a triple $(\tau, h_*, p)$, with $\tau$ a morphism in $\mathcal{C}$, $h_* = (h_1 \leq \ldots \leq h_s)$, and $p$ as in (4.2.4.1), we denote the morphism given by the composition (4.2.4.1) by $q(\tau, h_*, p)$. We let

$$\bar{q}(\tau, h_*, p): X \rightarrow Y$$

(4.2.4.2)
be the composition

$$X \overset{p}{\to} W_1 \times_S \ldots \times_S W_s \times_S Y \overset{p_Y}{\to} Y.$$

**(4.2.5) Lemma**

i) The map (4.2.4.2) depends only on $q(\tau, h_*, p)$, not on the choice of $\tau$, $h_*$ and $p$.

ii) when defined, the composition

$$q(\tau_2, h_{*2}, p_2) \circ q(\tau_1, h_{*1}, p_1)$$

is a map of the form $q(\tau, h_*, p)$, or is zero; if the composition is not zero, then

$$\bar{q}(\tau_2, h_{*2}, p_2) \circ \bar{q}(\tau_1, h_{*1}, p_1) = q(\tau, h_*, p).$$

**Proof.** By (II, (1.7.2) and (2.4.9)), the ambiguity in the choice of $(\tau; h_*, p)$ is given by

$$q(\pm \sigma \circ \tau, h_*, p) = \pm q(\tau, h_*, (t_\sigma \times \text{id}_Y) \circ p)$$

where $\sigma$ is in $S(h_*) \subset S_s$, and

$$t_\sigma : W_1 \times_S \ldots \times_S W_s \to W_{\sigma^{-1}(1)} \times_S \ldots \times_S W_{\sigma^{-1}(s)}$$

is the corresponding symmetry isomorphism. As

$$p_Y \circ (t_\sigma \times \text{id}_Y) = p_Y,$$

(i) is proven.

For (ii), suppose the $\tau_i$ are maps

$$\tau_i : \mathfrak{e}^{\otimes a_i} \to \mathfrak{e}^{\otimes a_i[\tau_i]}.$$

Using the notation of (4.2.4.1), in order that the composition in (ii) is defined, we have

$$a'_1 = a_2.$$

Let $\sigma$ be the shuffle permutation which puts the sequence $h_{*2}h_{*1}$ in increasing order, and let $h_*$ be the resulting increasing sequence. Let

$$W^i = W^i_1 \times_S \ldots \times_S W^i_{s'}, \quad i = 1, 2,$$

and let $W$ be the re-ordered version of $W^1 \times_S W^2$:

$$W = (W^1 \times_S W^2)^\sigma.$$
Let \( p: X_2 \to W \times Y_1 \)

be the composition

\[
X_2 \xrightarrow{p_2} W^2 \times Y_2 = W^2 \times X_1 \xrightarrow{id_{W^2} \times p_1} W^2 \times W^1 \times Y_1 \xrightarrow{t_{a_1}^{-1} \times id_Y} W \times Y_1.
\]

Then

\[
q(\tau_2, h_{*2}, p_2) \circ q(\tau_1, h_{*1}, p_1) = \pm q(\sigma \circ (\tau_2 \otimes id_{\epsilon \otimes a_{1-2}}) \circ \tau_1, h_*, p).
\]

As

\[
p_{Y_1} \circ p = (p_{Y_1} \circ p_1) \circ (p_{Y_2} \circ p_2),
\]

the proof of (ii) is complete. \( \square \)

(4.2.6) Zariski open covers for the category \( \mathcal{A}_{mot} \)

Let \( X \) and \( Y \) be in \( \mathcal{V} \), and let

\[
\mathcal{U} = \{ U_0, \ldots, U_m \}
\]

be a Zariski open cover of \( Y \). Suppose we have a map

\[
q: \mathcal{C}^{\otimes a} \otimes \mathbb{Z}_Y(b) [c] \to \mathcal{C}^{\otimes a'} \otimes \mathbb{Z}_X(b') [c'].
\]

We now define the Zariski open cover

\[
q_* \mathcal{U}
\]

of \( X \), and a commutative diagram in \( \mathbb{C}^{b}_{mot}(\mathcal{V})^* \):

\[
\begin{array}{ccc}
\mathcal{C}^{\otimes a} \otimes \mathbb{Z}_Y(b) [c] & \xrightarrow{j_{Y, \mathcal{U}}} & \mathcal{C}^{\otimes a} \otimes \mathbb{Z}_{U_{b,a}} (b) [c] \\
\downarrow q & & \downarrow q_{\mathcal{U}} \\
\mathcal{C}^{\otimes a'} \otimes \mathbb{Z}_X(b') [c'] & \xrightarrow{j_{X,q_* \mathcal{U}}} & \mathcal{C}^{\otimes a'} \otimes \mathbb{Z}_{q_* U_{b,a}} (b') [c'].
\end{array}
\] (4.2.6.1)

Suppose at first that \( X \) and \( Y \) are connected, and that \( q = q(\tau, h_*, p) \), giving the map

(4.2.4.2)

\[
\bar{q} := \bar{q}(\tau, h_*, p): X \to Y.
\]

For an open subset \( U \) of \( Y \), we have the open subset \( \bar{q}^{-1}(U) \) of \( X \).

For each open subscheme \( j_U: U \to Y \), let

\[
k_V: V \to X
\]
denote the inclusion of \( \overline{q}^{-1}(U) \) into \( X \). If \( j_{U,U'}: U' \to U \) is the inclusion of open subschemes of \( Y \), we have the induced inclusion

\[
k_{V,V'}: V' \to V
\]

where \( V' = \overline{q}^{-1}(U') \). We have the map

\[
q_{U'}: e^{\otimes a} \otimes Z_U(b)_{j_{U'}^*}g[c] \to e^{\otimes a'} \otimes Z_V(b')_{k_{V'}^*}f[c'] \tag{4.2.6.2}
\]
defined as the composition

\[
e^{\otimes a} \otimes Z_U(b)_{j_{U'}^*}g[c] \xrightarrow{\tau \otimes \text{id}_Y} e^{\otimes a} \otimes Z_U(b)_{j_{U'}^*}g[c + r] = e^{\otimes a'} \otimes e^{\otimes a-a'} \otimes Z_U(b)_{j_{U'}^*}g[c + r]
\]

\[
\xrightarrow{\text{id}_Y \otimes h_1 \otimes \ldots \otimes h_s \otimes \text{id}_Y} e^{\otimes a'} \otimes Z_{W_1 \times \ldots \times W_s \times U}(\sum_i b_i + b)_{j_{W_1 \times \ldots \times W_s \times U}^*}g[c + r]
\]

\[
\xrightarrow{\text{id}_Y \otimes p_{U'}^*} e^{\otimes a'} \otimes Z_{V'}(b')_{k_{V'}^*}f[c']
\]

The functoriality of the external products \( e_{**} \) implies the commutativity of

\[
\begin{array}{ccc}
e^{\otimes a} \otimes Z_U(b)_{j_{U'}^*}g[c] & \xrightarrow{q_{U'}} & e^{\otimes a'} \otimes Z_V(b')_{k_{V'}^*}f[c'] \\
j_{U,U'}^* & & \downarrow k_{V,V'}^* \\
e^{\otimes a} \otimes Z_U(b)_{j_{U'}^*}g[c] & \xrightarrow{q_{U'}} & e^{\otimes a'} \otimes Z_V(b')_{k_{V'}^*}f[c']
\end{array}
\tag{4.2.6.3}
\]

for \( U' \subseteq U \), with \( V = \overline{q}^{-1}(U) \), \( V' = \overline{q}^{-1}(U') \).

If \( U = \{U_0, \ldots, U_m\} \) is an open cover of \( Y \), we let \( q_*U \) be the open cover of \( X \) defined by

\[
q_*U = \{\overline{q}^{-1}(U_0), \ldots, \overline{q}^{-1}(U_m)\}
\]

Let \( V_i = \overline{q}^{-1}(U_i) \).

Let \( I = (i_0 < \ldots < i_s) \), and let

\[
j_I: U_I \to Y; \quad k_I: V_I \to X
\]

be the inclusions; for \( I \subseteq J \), we have the inclusions

\[
j_{I \subseteq J}: U_J \to U_I; \quad k_{I \subseteq J}: V_J \to V_I.
\]

Using the commutativity of the diagram (4.2.6.3), the collection of maps \( q_I \) defines the map

\[
q_{U^*}: e^{\otimes a} \otimes Z_{U^*}^{\text{op}}(b)_g[c] \to e^{\otimes a'} \otimes Z_{q_*U^*}(b')_f[c'] \tag{4.2.6.4}
\]
Bloch’s higher chow groups

giving the desired commutative diagram (4.2.6.1).

Suppose $q$ is a sum of compositions (4.2.4.1)

$$q = \sum_{i=1}^{m} n_i q_i; \quad n_i \in \mathbb{Z}, n_i \neq 0.$$  

with the $q_i$ basis elements as described in (4.2.4). Order the $q_i$ so that the maps

$$\tilde{q}_1, \ldots, \tilde{q}_s$$

are distinct, and so that

$$\{\tilde{q}_1, \ldots, \tilde{q}_s\} = \{\tilde{q}_1, \ldots, \tilde{q}_m\}.$$  

Let $q_* \mathcal{U}$ be cover given by the open subsets

$$\tilde{q}_1^{-1}(U_{i_1}) \cap \ldots \cap \tilde{q}_s^{-1}(U_{i_s});$$

we have the canonical refinement maps for each $i$

$$\rho_i: q_* \mathcal{U} \rightarrow q_{i*} \mathcal{U}.$$  

Forming the maps $q_{id\mathcal{U}}$ (4.2.6.4) for each $i$, composing with the refinement map $\rho_i^*$, and summing, gives the desired map

$$q_{\mathcal{U}}: \mathcal{C}^a \otimes \mathbb{Z}_{\mathcal{U}_{\nu-a}}(b)_{[\mathcal{C}]} \rightarrow \mathcal{C}^{a'} \otimes \mathbb{Z}_{q_* \mathcal{U}_{\nu-a}}(b')_{[c]},$$

$$q_{\mathcal{U}} = \sum_{i=1}^{m} n_i (\rho_i^* \circ q_{i,*} \mathcal{U}). \quad (4.2.6.5)$$

If $q$ is the zero map, we define $q_* \mathcal{U}$ to be the trivial cover $X$, and $q_{\mathcal{U}}$ to be the zero map.

The formation of $q_{\mathcal{U}}$ and $q_* \mathcal{U}$ is compatible with refinement: each refinement $\rho: \mathcal{V} \rightarrow \mathcal{U}$ gives the refinement

$$q_* \rho: q_* \mathcal{V} \rightarrow q_* \mathcal{U};$$

and we have the identity

$$(q_* \rho)^* \circ q_{\mathcal{U}} = q_{\rho^* \mathcal{U}} \circ \rho^* \quad (4.2.6.6)$$

We may not have the identities

$$q'_* q_* \mathcal{U} = (q' \circ q)_* \mathcal{U}; \quad (q' \circ q)_{\mathcal{U}} = q'_{q_* \mathcal{U}} \circ q_{\mathcal{U}}$$

due to possible cancellations in the expression for $q' \circ q$, as well as identities among the compositions of the maps $\tilde{q}_i$ and $\tilde{q}_j$; however, it follows from (4.2.5) that there is a refinement

$$\rho_{q', q_*} q_* q_{\mathcal{U}} \rightarrow (q' \circ q)_* \mathcal{U}$$
For similar reasons, there is a refinement
\[ \rho_{D, q}: q_* \mathcal{U} \to (Dq)_* \mathcal{U} \]
and we have the relation
\[ D_1(q_\mathcal{U}) = \rho_{D, q}^* \circ (Dq)_\mathcal{U}, \]
(4.2.6.7)
where \( D_1 \) refers to the differential with respect to the category \( \mathcal{A}_{mot}(\mathcal{V})^* \), not the Čech differential.

We extend the definition of \( q_* \mathcal{U} \), \( q'_* \mathcal{U} \) and \( \rho_{q', q} \) to arbitrary objects of \( \mathcal{A}_{mot}(\mathcal{V})^* \) by taking direct sums. The relations (4.2.6.6)-(4.2.6.7) continue to hold.

(4.2.7)

We recall from (II, (2.1.3)), that for a DG category \( \mathcal{A} \), we have the DG category \( \text{Pre-Tr}(\mathcal{A}) \) with objects \( X \) being tuples of the form \( (X_N, X_{N+1}, \ldots, X_M; q_{ij}) \), where \( N \leq M \) are integers, the \( q_{ij} \) are morphisms
\[ q_{ij}: X_j[-j] \to X_i[-i] \]
in \( \mathcal{A} \), and
\[ \sum_k q_{ik} \circ q_{kj} = Dq_{ij} \]
for all \( i \) and \( j \) (including \( i = j \)). There is an operation of Cone in \( \text{Pre-Tr}(\mathcal{A}) \), and the category \( \mathcal{C}^b(\mathcal{A}) \) is the smallest full DG subcategory of \( \text{Pre-Tr}(\mathcal{A}) \) containing \( \mathcal{A} \) and closed under taking Cones. If \( \mathcal{A} \) has trivial differential graded structure, \( \mathcal{C}^b(\mathcal{A}) \) is the usual DG category of bounded complexes in \( \mathcal{A} \). The operation
\[ (X_N, X_{N+1}, \ldots, X_M; q_{ij}) \mapsto \oplus_{i=N}^{M} X_i[-i] \]
defines the “forgetful functor”
\[ FD: \mathcal{C}^b(\mathcal{A}) \to \mathcal{A}; \]
in case \( \mathcal{A} \) has trivial differential structure, this is just the functor “forget the differential”. In particular, if we take the graded category \( Z^* \mathcal{A} \) with the same objects as \( \mathcal{A} \), where \( \text{Hom}_{Z^* \mathcal{A}}(X, Y) \) is the graded group of morphisms \( f: X \to Y \) in \( \mathcal{A} \) with \( Df = 0 \), the category \( \mathcal{C}^b(Z^* \mathcal{A}) \) is the category of bounded complexes in the degree zero portion \( Z^0 \mathcal{A} \) of \( Z^* \mathcal{A} \).

We recall from (II, (2.1.4) and (2.1.5)), that taking the total complex of a complex defines the functor (II, (2.1.4.1))
\[ \text{Tot}: \mathcal{C}^b(\mathcal{C}^b_{mot}(\mathcal{V}))^* \to \mathcal{C}^b_{mot}(\mathcal{V})^*. \]

(4.2.8) Definition
i) Let $\Gamma$ be an object of $\mathcal{C}^b_{mot}(\mathcal{V})^*$. A Zariski open cover of $\Gamma$ is a Zariski open cover of $FD(\Gamma)$, a refinement of a Zariski open cover of $\Gamma$ is a refinement of the corresponding Zariski open cover of $FD(\Gamma)$.

ii) Suppose we have

$$\Gamma = (\Gamma_N, \ldots, \Gamma_M; q_{ij})$$

for objects $\Gamma_i$ of $\mathcal{A}_{mot}(\mathcal{V})^*$ and maps $q_{ij}: \Gamma_j[-j] \to \Gamma_i[-i]$ in $\mathcal{A}_{mot}(\mathcal{V})^*$. A Čech resolution of $\Gamma$ is a map

$$j: \Gamma \to \Gamma_U$$

in $\mathcal{C}^b(Z^0C^b_{mot}(\mathcal{A})^*)$ such that

a) There are Zariski open covers $\mathcal{U}_i$ of $\Gamma_i$, and Čech resolutions

$$j_i := j_{\Gamma_i, \mathcal{U}_i}: \Gamma_i \to (\Gamma_i)_{\mathcal{U}_i}$$

with associated open cover $\mathcal{U}_i$, $i = N, \ldots, M$.

b) for each $i$ and $j$ with $q_{ij} \neq 0$, there is a refinement (on $\Gamma_i$)

$$\rho_{ij}: \mathcal{U}_i \to q_{ij}\mathcal{U}_j$$

c)

$$\Gamma_U = ((\Gamma_N)_{\mathcal{U}_N}, \ldots, (\Gamma_M)_{\mathcal{U}_M}; \tilde{q}_{ij})$$

$$\tilde{q}_{ij}: (\Gamma_j)_{\mathcal{U}_j}[-j] \to (\Gamma_i)_{\mathcal{U}_i}[-i]$$

with

$$\tilde{q}_{ij} = \rho_{ij}^* \circ (q_{ij})_{\mathcal{U}_i}$$

Letting $\mathcal{U}$ be the Zariski open cover of $\Gamma$ determined by the $\mathcal{U}_i$, we say that the Čech resolution $j$ has associated cover $\mathcal{U}$.

iii) A map of Čech resolutions

$$\tilde{q}: (j: \Gamma \to \Gamma_U) \to (j': \Gamma' \to \Gamma'_{U'})$$

over a map

$$q: \Gamma \to \Gamma'$$

in $Z^0C^b_{mot}(\mathcal{V})^*$ is a map

$$\tilde{q}: \Gamma_U \to \Gamma'_{U'}$$

in $\mathcal{C}^b(Z^0C^b_{mot}(\mathcal{V})^*)$ such that

$$\tilde{q} \circ j = j' \circ \tilde{q}$$

and such that the map $FD(\tilde{q})$ is a map of the form

$$\rho_{U', q_{U'}}^* \circ q_{U}$$
where $\mathcal{U}$ and $\mathcal{U}'$ are the open covers of $\Gamma$ and $\Gamma'$ corresponding to $j$ and $j'$, for some choice of refinement mapping $\rho_{\mathcal{U}'}, q_*, \mathcal{U}' \to q_* \mathcal{U}$.

iii) A sequence of maps

\[ j_0: \Gamma \to \Gamma_{U_1} \]
\[ j_1: \text{Tot}(\Gamma_{U_1}) \to \Gamma_{U_2} \]
\[ \vdots \]
\[ j_{m-1}: \text{Tot}(\Gamma_{U_{m-1}}) \to \Gamma_{U_m} \]

in $\mathcal{C}^b(Z^0 \mathcal{C}_{mot}^b(\mathcal{A})^*)$ is called a length $m$ tower of Čech resolutions of $\Gamma$ if each map $j_i$ is a Čech resolution. A map of length $m$ towers over a map $q: \Gamma \to \Delta$ is a sequence of maps

\[ \tilde{q}_0: \Gamma_{U_i} \to \Delta_{W_i} \]

such that $\tilde{q}_1$ is a map of Čech resolutions over $q$, and $\tilde{q}_{i+1}$ is a map of Čech resolutions over $\text{Tot}(\tilde{q}_i)$ for $1 \leq i \leq m - 1$. We often write a tower of Čech resolutions as

\[ \Gamma \xrightarrow{j_n} \Gamma_{U_n} \xrightarrow{j_{n-1}} \cdots \xrightarrow{j_0} \Gamma_{U_0}. \]

iv) If we have a tower of Čech resolutions of $\Gamma$ as in (iii), we call the composition

\[ j = \text{Tot} j_{m-1} \circ \ldots \circ \text{Tot} j_0: \Gamma \to \text{Tot} \Gamma_{U_m} \]

a hyper-resolution of $\Gamma$. Given two hyper-resolutions of $\Gamma$

\[ j: \Gamma \to \Gamma_{U_m}; \quad j': \Gamma' \to \Gamma'_{U'_m} \]

and a map

\[ f: \Gamma \to \Gamma' \]

in $Z^0 \mathcal{C}_{mot}^b(\mathcal{A})^*$, a map

\[ \tilde{f}: \Gamma_{U_m} \to \Gamma'_{U'_m} \]

in $\mathcal{C}^b(Z^0 \mathcal{C}_{mot}^b(\mathcal{A})^*)$ is a map of hyper-resolutions over $f$ if there is an $m$ and a map over $f$, $f_* =: (f_1, \ldots, f_m)$, of length $m$ towers of Čech resolutions, such that $\tilde{f}$ is the map $f_m$. A map of hyper-resolutions of $\Gamma$ is a map of hyper-resolutions over $\text{id}_\Gamma$.

We let $\mathbf{HR}$ be the subcategory of $\mathcal{C}^b(Z^0 \mathcal{C}_{mot}^b(\mathcal{V})^*)$ with objects the hyper-resolutions of objects of $\mathcal{C}^b_{mot}^b(\mathcal{V})^*$, and maps the maps of hyper-resolutions; we let $\mathbf{HR}_\Gamma$ be the subcategory of $\mathbf{HR}$ with objects the hyper-resolutions of $\Gamma$ and maps being maps over the identity.
Remark

It follows directly from (4.2.8) that, if

\[ \tilde{q}: (j: \Gamma \to \Gamma_U) \to (j': \Gamma' \to \Gamma'_{U'}) \]

is a map of Čech resolutions over a map

\[ q: \Gamma \to \Gamma' \]

in \( Z^0 C^b_{mot}(\mathcal{V})^* \), then

\[ (j[1], j'): \text{Cone}(q) \to \text{Cone}(\tilde{q}) \]

is a Čech resolution of \( \text{Cone}(q) \), giving the commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{j} & \Gamma_U \\
q \downarrow & & \downarrow \tilde{q} \\
\Gamma' & \xrightarrow{j'} & \Gamma'_{U'} \\
\downarrow & & \downarrow \\
\text{Cone}(q) & \xrightarrow{(j[1], j')} & \text{Cone}(\tilde{q}) \\
\downarrow & & \downarrow \\
\Gamma[1] & \xrightarrow{j[1]} & \Gamma_U[1]
\end{array}
\]

with the columns standard Čech sequences.

Lemma

i) Let \( \Gamma \) be in \( C^b_{mot}(\mathcal{V})^* \), \( \mathcal{U} \) a Zariski open cover of \( \Gamma \). Then there is a refinement \( \mathcal{W} \to \mathcal{U} \) of \( \mathcal{U} \) and a Čech resolution

\[ j: \Gamma \to \Gamma_{\mathcal{W}} \]

of \( \Gamma \) with associated cover \( \mathcal{W} \).

ii) Suppose we have a map \( q: \Gamma \to \Gamma' \) in \( Z^0 C^b_{mot}(\mathcal{V})^* \), and Čech resolutions

\[ j: \Gamma \to \Gamma_{\mathcal{W}}; \quad j': \Gamma' \to \Gamma'_{\mathcal{W}'} \]

with associated covers \( \mathcal{W} \) and \( \mathcal{W}' \). Then there is a refinement \( \mathcal{U}' \) of \( \mathcal{W}' \), a Čech resolution

\[ j'': \Gamma' \to \Gamma'_{\mathcal{U}'} \]

with associated cover \( \mathcal{U}' \), a map of Čech resolutions

\[ \tilde{q}: \Gamma_{\mathcal{W}} \to \Gamma_{\mathcal{U}'} \]
over $q$ and a map of Čech resolutions over the identity

$$\text{id}: \Gamma'_{\mathcal{V}} \to \Gamma_{\mathcal{U}}$$

(iii) Let $\Gamma$ be in $\mathbf{C}^b_{mot}(\mathcal{V})^*$, and let

$$j: \Gamma \to \Gamma_{\mathcal{U}}$$

be a Čech resolution of $\Gamma$. Then

$$\text{Tot}(j): \Gamma \to \text{Tot}(\Gamma_{\mathcal{U}})$$

is an isomorphism in $\mathbf{D}^b_{mot}(\mathcal{V})$.

Proof. Let

$$\Gamma = (\Gamma_N, \ldots, \Gamma_M; q_{ij})$$

be in $\mathbf{C}^b_{mot}(\mathcal{V})^*$. As $\Gamma$ is an iterated Cone of objects of $\mathbf{A}_{mot}(\mathcal{V})^*$, we may write each $\Gamma_i$ as a direct sum

$$\Gamma_i = \bigoplus_k \Gamma_{ik}$$

such that the component

$$q^{kk'}_{ij}: \Gamma_{jk'}[-j] \to \Gamma_{ik}[-i]$$

is zero if $k \leq k'$. Let $\Gamma^k = \bigoplus_i \Gamma_{ik}[-i]$; we then have the collection of maps

$$q^{kk'}: \Gamma^k \to \Gamma^{k'}; \quad N_1 \leq k' < k \leq M_1$$

Thus, given a Zariski open cover $\mathcal{U}$ of $\Gamma$, we have for each $k$ a Zariski open cover $\mathcal{U}_k$ of $\Gamma^k$ for all $k$. We may then inductively choose refinements $\mathcal{W}_k$ of $\mathcal{U}_k$ so that $\mathcal{W}_k$ is a refinement of $q_*^{kk'} \mathcal{W}_{k'}$ and for each $k' < k$ via

$$\rho^{kk'}: \mathcal{W}_k \to q_*^{kk'} \mathcal{W}_{k'}$$

Let $\mathcal{W}_{ik}$ be the restriction of $\mathcal{W}_k$ to $\Gamma_{ik}$, let $\rho^{kk'}_{ij}$ be the restriction of $\rho^{kk'}$ to the refinement

$$\rho^{kk'}_{ij}: \mathcal{W}_{ik} \to q_*^{kk'} \mathcal{W}_{jk'}$$

and let

$$\tilde{q}^{kk'}_{ij}: (\Gamma_{jk'}[-j])_{\mathcal{W}_{jk'}} \to (\Gamma_{ik}[-i])_{\mathcal{W}_{ik}}$$

be the map

$$\tilde{q}^{kk'}_{ij} = (\rho^{kk'}_{ij})^* \circ (q^{kk'}_{ij})_{\mathcal{W}_{jk'}}$$

The relations (4.2.6.6)-(4.2.6.7) imply that the tuple

$$\left(\ldots, \bigoplus_k (\Gamma_{ik})_{\mathcal{W}_{ik}}, \ldots; \bigoplus_k \tilde{q}^{kk'}_{ij}\right)$$
defines an object $\Gamma_W$ of $C^b(Z^0C_{mot}^b(V))$, and that the collection of maps

$$j_{W_{ik}} : \Gamma_{ik} \to (\Gamma_{ik}[-i])_{W_{ik}}$$

define a Čech resolution

$$j_W : \Gamma \to \Gamma_W.$$  

This proves (i).

If we already have a Čech resolution for the cover $U$,

$$j_U : \Gamma \to \Gamma_U$$

then the maps in $\Gamma_U$ are completely determined by the maps in $\Gamma$ and the choice of refinement mappings

$$\rho_{ij}^U : U_i \to q_{ij}U_j.$$  

Thus, if we have any Zariski open cover $W$ of $\Gamma$, using the inductive procedure as above, we may find a common refinement $W'$ of $W$ and $U$:

$$\nu_{U,W'} : W' \to U$$

such that there are refinements

$$\rho_{ij}' : W'_i \to q_{ij}W'_j$$

with

$$q_{ij}\nu_{U_i,W'_j} \circ \rho_{ij}' = \rho_{ij}^U \circ \nu_{U_i,W_i}.$$  

This then implies that we may form the Čech resolution of $\Gamma$ as constructed above for the cover $W'$ together with the choice of refinement maps $\rho_{ij}'$:

$$j_{W'} : \Gamma \to \Gamma_{W'}$$

and that the refinement maps

$$\nu_{U_i,W'_i}^* : \Gamma_{iU_i} \to \Gamma_{iW'_i}$$

define a map of Čech resolutions

$$\nu_{U_i,W'_i}^* : \Gamma_U \to \Gamma_{W'_i}$$

over the identity. To finish the proof of (ii), we apply the construction of (i) to $\text{Cone}(q)$, and note that the resulting Čech resolution is the Cone of a map $\tilde{q}$ of Čech resolutions over the map $q$.

For (iii), we have already remarked (4.2.2.2) that $j$ is an isomorphism in $D^b_{mot}(V)$ in case $\Gamma$ is an object of the form $e^{\otimes a} \otimes \mathbb{Z}_X(q)_f[p]$. As such objects generate $A_{mot}(V)^*$ as
an additive category, $j$ is an isomorphism in $\mathbf{D}^b_{\operatorname{mot}}(\mathcal{V})$ for all $\Gamma$ in $\mathcal{A}_{\operatorname{mot}}(\mathcal{V})^*$. In general, suppose $\Gamma$ is a cone:

$$\Gamma_0 \xrightarrow{q} \Gamma_1 \longrightarrow \Gamma = \operatorname{Cone}(q) \longrightarrow \Gamma_0[1].$$

By definition of a Čech resolution, we may fit the Čech resolution

$$j: \Gamma \to \Gamma_{U}$$

in a commutative diagram

$$\begin{array}{cccc}
\Gamma_{U_0} & \xrightarrow{q} & \Gamma_{U_1} & \longrightarrow & \Gamma_{U} = \operatorname{Cone}(\tilde{q}) & \longrightarrow & \Gamma_{U_0}[1] \\
\downarrow{j_0} & & \downarrow{j_1} & & \downarrow{j} & & \downarrow{j_0[1]} \\
\Gamma_0 & \xrightarrow{q} & \Gamma_1 & \longrightarrow & \Gamma = \operatorname{Cone}(q) & \longrightarrow & \Gamma_0[1]
\end{array}$$

with both rows cone sequences. As $\mathbf{C}^b_{\operatorname{mot}}(\mathcal{V})^*$ is generated by $\mathcal{A}_{\operatorname{mot}}(\mathcal{V})^*$ by taking cones, this proves (iii). □

(4.2.11) **Proposition**

Let $\Gamma$ be in $\mathbf{C}^b_{\operatorname{mot}}(\mathcal{V})^*$. Then the image of the category $\mathbf{HR}_\Gamma$ in the homotopy category $\mathbf{K}^b(Z^0\mathbf{C}^b_{\operatorname{mot}}(\mathcal{V}))$ is left-filtering.

**Proof.** As the identity map on $\Gamma$ is a hyper-resolution of $\Gamma$, $\mathbf{HR}_\Gamma$ is non-empty. In addition, if we have a length $m$ tower of Čech resolutions of $\Gamma$, we may extend the tower to length $m+1$ by adjoining an identity map at the end of the tower, so we may always consider two hyper-resolutions of $\Gamma$ as coming from towers of Čech resolutions of $\Gamma$ of the same length. Thus, it follows from (4.2.10)(ii) that, given two hyper-resolutions of $\Gamma$

$$j: \Gamma \to \tilde{\Gamma}; \quad j': \Gamma \to \tilde{\Gamma}',$$

there is a hyper-resolution $j'': \Gamma \to \tilde{\Gamma}''$ and maps of hyper-resolutions

$$\tilde{\Gamma}'' \to \tilde{\Gamma}; \quad \tilde{\Gamma}'' \to \tilde{\Gamma}'.$$

Now suppose we have two maps of hyper-resolutions of $\Gamma$

$$\tilde{f}^1, \tilde{f}^2: \tilde{\Gamma} \to \tilde{\Gamma}'.$$

As above, we may assume that the maps $\tilde{f}^1$, $\tilde{f}^2$ come from maps of length $m$ towers of Čech resolutions

$$f^1_*, f^2_*: \Gamma_* \to \Gamma'_*.$$  

From (4.2.8), the only choice that one has in forming a map of towers of Čech resolutions is the choice of the various refinement mappings, which must satisfy the compatibility requirement of (4.2.8)(ii)(c).
As is well known, if we have open covers
\[ U = \{U_0, \ldots, U_m\}; \quad V = \{V_0, \ldots, V_n\} \]
are open covers and if we have two refinement mappings
\[ \rho_1, \rho_2: V \to U \]
\[ V_i \subset U_{\rho_1(i)}; \quad V_i \subset U_{\rho_2(i)} \]
there is the functorial homotopy \( H \) on chain complexes associated to the maps of simplicial schemes
\[ N(\rho_1), N(\rho_2): NV \to NU \]
\[ N(\rho_j): V_{i_0} \cap \ldots \cap V_{i_k} \to U_{\rho_{j_i}(i_0)} \cap \ldots \cap U_{\rho_{j_i}(i_k)} \]
defined by sending \( V_{i_0} \cap \ldots \cap V_{i_k} \) to the sum
\[ \sum_{j=0}^{k} (-1)^j [U_{\rho_{1}(i_0)} \cap \ldots \cap U_{\rho_{1}(i_j)} \cap U_{\rho_{2}(i_j)} \cap \ldots \cap U_{\rho_{2}(i_k)}]. \]
Here
\[ [U_{j_0} \cap \ldots \cap U_{j_k}] = \text{sgn}(j_0, \ldots, j_k) \cdot (U_{j_0} \cap \ldots \cap U_{j_k}) \]
and \( \text{sgn}(j_0, \ldots, j_k) \) is the sign of the permutation which puts \( j_0, \ldots, j_k \) in increasing order if the \( j_i \) are distinct, and is zero if the \( j_i \) are not distinct.

Since the homotopy \( H \) is natural, we may apply \( H \) to the two compatible choices of refinement mappings determined by the maps \( f_1^* \) and \( f_2^* \), giving the desired homotopy between \( f^1 \) and \( f^2 \) in \( C^b(Z^0 C_{mot}^b(V)) \).

(4.2.12) Hypercohomology

Now let
\[ h: C^b_{mot}(V)^* \to C(Ab) \]
be an DG functor, compatible with Cones. We define the hypercohomology of \( \Gamma \) with respect to \( h \) as
\[ H^0_h(\Gamma) := \lim_{\text{ulimit}} H^0(h(Tot(\Gamma_{\tilde{\mu}})) \quad (4.2.12.1) \]
where the limit is over the category of hyper-resolutions of \( \Gamma \). By (4.2.11), the limit in (4.2.12.1) is equivalent to a filtered direct limit. We have the natural map
\[ h(\Gamma) \to H^0_h(\Gamma), \]
hence sending $\Gamma$ to $\mathbb{H}_h^0(\Gamma)$ defines a cohomological functor and exact natural transformation

$$
\mathbb{H}_h^0 : \mathbb{K}_{mot}(\mathcal{V})^* \to \text{Ab},
\mathbb{H}^0 : h \to \mathbb{H}_h^0.
$$

We have the DG functor (4.1.10.1), compatible with Cones

$$
\mathcal{Z}_{mot}(\ast) : \mathbb{C}_{mot}(\mathcal{V})^* \to \mathcal{C}^{-}(\text{Ab})
\Gamma \mapsto \mathcal{Z}_{mot}(\Gamma, \ast).
$$

For each $N$, we have the DG functor, compatible with Cones

$$
\Sigma^N \mathcal{Z}_{mot}[N] : \mathbb{C}_{mot}(\mathcal{V})^* \to \mathcal{C}^{-}(\text{Ab})
$$

(4.2.12.3)

defined as the composition

$$
\Sigma^N \mathcal{Z}_{mot}[N] := \mathcal{Z}_{mot} \circ \Sigma^N(-)[N]
$$

We have the natural transformation (4.1.10.4)

$$
\Pi_N : \Sigma^N \mathcal{Z}_{mot}[N] \to \mathcal{Z}_{mot}(\ast),
$$

(4.2.12.4)

inducing the natural transformation

$$
\mathbb{H}^0(\Pi_N) : \mathbb{H}_\Sigma^0 \mathcal{Z}_{mot}[N] \to \mathbb{H}_\mathcal{Z}_{mot}(\ast)
$$

(4.2.12.5)

**Definition**

Let $\Gamma$ be in $\mathbb{C}_{mot}(\mathcal{V})^*$. Define the higher Chow groups of $\Gamma$ by

$$
\mathcal{C}H(\Gamma, p) = \mathbb{H}_\mathcal{Z}_{mot}(\ast)(\Gamma[-p])
$$

(cf. (4.2.12.1)). We write $\mathcal{C}H(\Gamma)$ for $\mathcal{C}H(\Gamma, 0)$. The natural transformation

$$
\mathbb{H}^0 : \mathcal{Z}_{mot} \to \mathbb{H}_\mathcal{Z}_{mot}(\ast)
$$

gives the natural map

$$
\text{CH}_{\text{naive}}(\Gamma, p) \to \mathcal{C}H(\Gamma, p).
$$

Before proceeding further with the Chow groups, we prove some results relating the functor $\mathbb{H}_h^*$ with Zariski hypercohomology.
(4.2.14) Lemma

i) Let \( h: C^{b}_{mot}(\mathcal{V})^* \to C^-(\text{Ab}) \); \( n = 0, 1, \ldots, \infty \)
be a DG functor, compatible with Cones, and let \( (X, f) \) be in \( \mathcal{L}(\mathcal{V}) \). Let \( \tilde{h}^X \) be the complex of Zariski sheaves on \( X \) associated to the presheaf \( h^X \) given by

\[
(j: U \to X) \mapsto h(\mathcal{E}^{\otimes a} \otimes \mathbb{Z}_U(q)_{j^*f}[p])
\]

Let \( \mathbb{H}^{0}_{\text{Zar}}(X, \tilde{h}^X) \) denotes the Zariski hypercohomology. Then there is a natural isomorphism

\[
\mathbb{H}^{0}_{h}(\mathcal{E}^{\otimes a} \otimes \mathbb{Z}_X(q)_{f}[p]) \cong \mathbb{H}^{0}_{\text{Zar}}(X, \tilde{h}^X).
\]

ii) Let \( h_n: C^{b}_{mot}(\mathcal{V})^* \to C^-(\text{Ab}) \); \( n = 0, 1, \ldots, \infty \)
be DG functors, compatible with Cones, together with a sequence of natural transformations

\[
h_0 \xrightarrow{\pi_{10}} h_1 \xrightarrow{\pi_{21}} \ldots
\]

and natural transformations

\[
\pi_n: h_n \to h_\infty,
\]

compatible with Cones, such that

\[
\pi_{n+1} \circ \pi_{n+1,n} = \pi_n.
\]

Suppose that, for each pair of integers \( p \) and \( q \), there is an integer \( N_{p,q} \) such that

\[
H^0(p_n): H^0(h_n(\mathbb{Z}_X(q)_{f}[m])) \to H^0(h_\infty(\mathbb{Z}_X(q)_{f}[m]))
\]

is an isomorphism for all \( (X, f) \) in \( \mathcal{L}(\mathcal{V}) \), and all \( m \geq p \). Then, for each \( \Gamma \) in \( C^{b}_{mot}(\mathcal{V})^* \), there is an integer \( N_\Gamma \) such that the map

\[
\mathbb{H}^0(p_N): \mathbb{H}^0_{h_N}(\Gamma) \to \mathbb{H}^0_{h_\infty}(\Gamma),
\]

is an isomorphism for all \( N \geq N_\Gamma \).

Proof. For (i), let \( j: U \to X \) be a Zariski open subset of \( X \). We set

\[
\Gamma_U = \mathcal{E}^{\otimes a} \otimes \mathbb{Z}_U(q)_{j^*f}[p]
\]

For a presheaf \( S \) on \( X \), we set

\[
S(\Gamma_U) = S(U);
\]

for an inclusion of open subsets \( k: V \to U \), we let

\[
S(id_{\mathcal{E}^{\otimes a} \otimes k^*}) = S(k).
\]
We extend these constructions to complexes of presheaves in the obvious way by taking the associated total complex.

If $S$ is a complex of presheaves, and

$$
\Gamma_U \rightarrow \Gamma_U
$$

is a Čech resolution, we have the natural isomorphism

$$
S(\Gamma_U) \rightarrow S(U)
$$

(1)

where $S(U)$ is the ordered non-degenerate Čech complex of $S$ associated to the cover $U$ of $U$. If $S$ is a sheaf, we thus have have

$$
H^0(S(\Gamma_U)) = S(U)
$$

(2)

and if $S$ is an injective sheaf, we have

$$
H^p(S(\Gamma_U)) = 0
$$

(3)

for $p > 0$.

Now let

$$
\Gamma_U \rightarrow \bar{\Gamma}
$$

(4)

be a hyper-resolution of $\Gamma_U$. By (1)-(3), induction and an elementary spectral sequence argument, we have

$$
H^0(S(\bar{\Gamma})) = S(U)
$$

(5)

if $S$ is a sheaf, and if $S$ is an injective sheaf, we have

$$
H^p(S(\bar{\Gamma})) = 0
$$

(6)

for $p > 0$.

We note that $h^X$ is an abelian presheaf on $X$, and we have

$$
h^X(\Gamma) = h(\Gamma)
$$

(7)

for each hyper-resolution (4) of $\Gamma_U$.

Let $\mathcal{P}$ be a presheaf on $X$ whose associated sheaf is zero, let $\bar{\Gamma}$ be a hyper-resolution of $\Gamma_X$, and let $\alpha$ be a degree $d$ element of $\mathcal{P}(\bar{\Gamma})$. From (4.2.10)(i), there is a map of hyper-resolutions of $X$

$$
f: \bar{\Gamma} \rightarrow \bar{\Delta}
$$

over the identity such that

$$
\mathcal{P}(f)(\alpha) = 0.
$$
From this and (4.2.11), it follows that

$$H^*(P) := \lim_{\tilde{\Gamma}} H^*(\mathcal{P}(\Gamma^*))$$

(where the limit is over hyper-resolutions $\tilde{\Gamma}$ of $\Gamma_X$) defines a $\delta$-functor on the category of sheaves on $X$.

It then follows from (5) that there is a canonical map of $\delta$-functors (on the category of sheaves on $X$)

$$H^*(-) \to H^*_{\text{Zar}}(X, -);$$

it follows from (6) that the map (8) is an isomorphism. In addition, we have

$$H^0(P) \to H^0_{\text{Zar}}(X, \tilde{P})$$

for a presheaf $P$ with associated sheaf $\tilde{P}$; from this and the isomorphism (9) we have the canonical isomorphism

$$H^*(P) \to H^*_\text{Zar}(X, \tilde{P})$$

This together with (7) proves (i) in case the functor $h$ maps $A_{mot}(V)$ into $\mathbf{Ab}$; the general case follows from this and a spectral sequence argument, noting that $X$ has finite cohomological dimension by ([G], Theorem 3.6.5).

We now prove (ii). The DG category $C^b_{mot}(V)^*$ is generated by $A_{mot}(V)^*$ by taking Cones. Since the functors $H^0_{h_n}$ are cohomological functors, it suffices to prove the result for $\Gamma$ in $A_{mot}(V)^*$. As this DG category is generated by the objects $\epsilon^{\otimes a} \otimes \mathbb{Z}_X(q)_f[p]$, it suffices to proof the result for $\Gamma$ of the form

$$\Gamma = \epsilon^{\otimes a} \otimes \mathbb{Z}_X(q)_f[p].$$

Let $\tilde{h}_n^X$ be the complex of Zariski sheaves on $X$ associated to the presheaf

$$(j: U \to X) \mapsto h_n(\epsilon^{\otimes a} \otimes \mathbb{Z}_U(q)_f[p])$$

Suppose $X$ has Krull dimension $M$. Then, by ([G], loc. cit.) for all Zariski sheaves $\mathcal{F}$ on $X$, we have

$$H^n(X, \mathcal{F}) = 0; \quad n > M.$$ 

By our assumption on the sequence of functors $h_n$, if $n \geq N_{p-M-1,q}$, the map of sheaves

$$\tilde{h}_n^X \to \tilde{h}_\infty^X$$

induces an isomorphism on the cohomology sheaves

$$\mathcal{H}^m(\tilde{h}_n^X) \to \mathcal{H}^m(\tilde{h}_\infty^X)$$
for all $m \geq -M - 1$. Applying this to the local to global spectral sequence for hypercohomology, we find that the natural map

$$H^0_{\text{Zar}}(X, \tilde{h}_n^X) \to H^0_{\text{Zar}}(X, \tilde{h}_\infty^X)$$

is an isomorphism for $n \geq N_{p-M-1,q}$. This, together with (i), completes the proof. □

(4.2.15) **Proposition**

Let $\Gamma$ be in $C^b_{\text{mot}}(V)^*$. Then there is an integer $N_\Gamma$ such that, for all $N \geq N_\Gamma$, the natural transformation (4.2.12.5) defines an isomorphism

$$H^0(\Pi_N)(\Gamma): H^0_{\Sigma^N Z_{\text{mot}}[N]}(\Gamma) \to \mathcal{C}(\Gamma).$$

**Proof.** This follows from (4.2.14) and (4.1.12). □

(4.2.16) **The refined cycle class map**

We now extend the naive cycle class map (4.1.15.1)

$$\text{cl}_{\text{naive}}(\Gamma): \text{CH}_{\text{naive}}(\Gamma) \to \text{Hom}_{D^b_{\text{mot}}(V)}(1, \Gamma)$$

to the cycle class map

$$\text{cl}(\Gamma): \mathcal{C}(\Gamma) \to \text{Hom}_{D^b_{\text{mot}}(V)}(1, \Gamma).$$

For this, let

$$j: \Gamma \to \Gamma_U$$

be a hyper-resolution of $\Gamma$. We have the naive cycle class map (4.1.15.1)

$$\text{cl}_{\text{naive}}(\text{Tot}(\Gamma_U)): \text{CH}_{\text{naive}}(\text{Tot}(\Gamma_U)) \to \text{Hom}_{D^b_{\text{mot}}(V)}(1, \text{Tot}(\Gamma_U));$$

as $j$ is an isomorphism in $D^b_{\text{mot}}(V)$ by (4.2.10)(iii), we may compose $\text{cl}_{\text{naive}}((\Gamma_U))$ with $j^{-1}$ giving the map

$$j^{-1} \circ \text{cl}_{\text{naive}}(\Gamma_U): \text{CH}_{\text{naive}}(\Gamma_U) \to \text{Hom}_{D^b_{\text{mot}}(V)}(1, \Gamma)$$

If we have another tower of Čech resolution of $\Gamma$, giving the hyper-resolution

$$j': \Gamma \to \Gamma_{U'}$$

and a map

$$\eta: \Gamma_U \to \Gamma_{U'}$$

over the identity, we have

$$j' = \eta \circ j,$$
as $\text{cl}_{\text{naive}}(-)$ is natural, the maps $j^{-1} \circ \text{cl}_{\text{naive}}(\Gamma_U)$ give a well-defined map on the limit

$$\text{cl}(\Gamma): \mathcal{CH}(\Gamma) \to \text{Hom}_{\mathcal{D}^{\text{mot}}(\mathcal{V})}(1, \Gamma)$$  \hspace{1cm} (4.2.16.1)

Via (4.2.10), we see that sending $\Gamma$ to $\text{cl}(\Gamma)$ defines a natural transformation of cohomological functors from $K^b_{\text{mot}}(\mathcal{V})$ to $\text{Ab}$.

\begin{definition}
Let $(X, f)$ be in $\mathcal{L}(\mathcal{V})$. We denote the complex $Z^q_{\text{mot}}(Z_X(q)[2q], *)$ by $\mathcal{Z}^q(X/S, *)_f$. We denote $\mathcal{CH}(Z_X(q)[2q - p])$ by $\mathcal{CH}^q(X/S, p)_f$, and the map

$$\text{cl}(Z_X(q)[p]): \mathcal{CH}(Z_X(q)[p])_f \to \text{Hom}_{\mathcal{D}^{\text{mot}}(\mathcal{V})}(1, Z_X(q)[2q - p]) = H^p(X, \mathbb{Z}(q))$$

by

$$\text{cl}^q_{X/S} : \mathcal{CH}^q(X/S, 2q - p)_f \to H^p(X, \mathbb{Z}(q)).$$

We write $\mathcal{CH}^q(X/S, p)$ for $\mathcal{CH}^q(X/S, p)_{\text{id}_X}$.

Sending $(X, f) \in \mathcal{L}(\mathcal{V})$ to $\mathcal{Z}^q(X/S, *)_f$ defines the functor

$$\mathcal{Z}^q(-/S, *)_f : \mathcal{L}(\mathcal{V})^{\text{op}} \to \text{C}^-(\text{Ab});$$

in particular, sending an open subscheme $j: U \to X$ to $\mathcal{Z}^q(U/S, *)_f$ defines a complex of presheaves on $X$; we let

$$\mathfrak{Z}^q_{X/S}(*)_f$$

denote the associated complex of Zariski sheaves.

\begin{equation}
\text{(4.2.18)}
\end{equation}

Let $\text{Sh}_{\text{Zar}, S}(\text{Ab})$ be the category of Zariski sheaves of abelian groups on $S$-schemes: an object is a sheaf $\mathcal{F}$ on an $S$-scheme $X$, and a morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(p, \tilde{p})$ consisting of a map $p: Y \to X$ and a map $\tilde{p}: \mathcal{F} \to p_*(\mathcal{G})$.

Sending $(X, f)$ to $\mathfrak{Z}^q_{X/S}(*)_f$ then gives the functor

$$\mathfrak{Z}^q_{X/S}(*)_f : \mathcal{L}(\mathcal{V})^{\text{op}} \to \text{C}^-(\text{Sh}_{\text{Zar}, S}(\text{Ab})), \hspace{1cm} (4.2.18.1)$$

where we send a morphism

$$p: (X, f) \to (Y, g)$$

to the pair $(p, p^*)$, where $p^*$ is the map

$$p^*: \mathfrak{Z}^q_{Y/S}(*)_g \to p_*(\mathfrak{Z}^q_{X/S}(*)_f).$$

By (4.2.14)(i), we have the canonical identification of $\mathcal{CH}^q(X/S, p)_f$ with the Zariski hypercohomology

$$\mathcal{CH}^q(X/S, p)_f \cong \mathbb{H}^{-p}_{\text{Zar}}(X, \mathfrak{Z}^q_{X/S}(*)_f).$$
We write $\mathcal{Z}^q_{X/S}(\ast)$ for $\mathcal{Z}^q_{X/S}(\ast)_{id_X}$.

Similarly, sending $(X, f)$ to $\mathcal{CH}^q(X/S, p)_f$ defines the functor

$$\mathcal{CH}^q(-, p)_f: \mathcal{L}(\mathcal{V})^{op} \to \textbf{Ab}$$

and the cycle class maps $\text{cl}^q_{X,p}$ define the natural transformation

$$\text{cl}^q_{X,p}: \mathcal{CH}^q(-, 2q - p)_f \to H^p(-, \mathbb{Z}(q)).$$

Let $i: Z \to X$ be a closed embedding of smooth $S$-schemes in $\mathcal{V}$ of relative codimension $d$. If $Y$ is in $\mathcal{V}$ and $W \in \mathcal{Z}^q-d(Z \times_S Y/S)$ is a cycle, we may consider $W$ as a cycle on $X \times_S Y$; this defines the natural transformation

$$i_*: \mathcal{Z}^{q-d}(Z \times_S (-)/S) \to \mathcal{Z}^q(X \times_S (-)/S).$$

This extends in the obvious way to a natural map of complexes

$$i_*: \mathcal{Z}^{q-d}(Z/S, \ast) \to \mathcal{Z}^q(X/S, \ast),$$

and to the natural map of complexes of sheaves on $X$:

$$i_*: i_* \mathcal{Z}^{q-d}(\ast) \to \mathcal{Z}^q_X(\ast).$$

If $j^*: U \to X$ is the complement $X \setminus Z$, we have

$$j^* \circ i_* = 0;$$

giving the natural map of complexes of sheaves on $X$:

$$i_*: i_* \mathcal{Z}^{q-d}(\ast) \to \text{Cone}(j^*: \mathcal{Z}^q_X(\ast) \to \mathcal{Z}^q_{U/S}(\ast))[-1] \quad (4.2.18.2)$$
4.3. Surjectivity of the cycle map

We give a general criterion for the surjectivity of the cycle map; we fix a localization $R$ of $\mathbb{Z}$ as coefficient ring.

**The surjectivity conditions**

For a complex of Zariski sheaves $\mathcal{F}$ of $R$-modules on a scheme $X$, we let $\mathcal{G}\mathcal{F}$ denote the total complex of the Godement resolution, and $\mathcal{R}_X \mathcal{F} = \mathcal{R}\mathcal{F}$ the global sections $\Gamma(X, \mathcal{G}\mathcal{F})$. If $\hat{X}$ is a closed subset of $X$, with complement $j: U \rightarrow X$, we let $\mathcal{R}_X^{\hat{X}} \mathcal{F}$ denote the Cone

$$\mathcal{R}_X^{\hat{X}} \mathcal{F} := \text{Cone}[\mathcal{R}^* j: \mathcal{R}_X \mathcal{F} \rightarrow \mathcal{R}_U (j^* \mathcal{F})][-1].$$

Consider the following conditions:

**The Chow realization**

We begin the construction of a functor from $\mathbf{D}^{b}_{\text{mot}}(\mathcal{V}) \otimes R$ to $\mathbf{D}^{b}(\text{Ab})$ which extends the assignment

$$X \mapsto \mathcal{R}_X \mathcal{Z}_{X/S}(\ast) \otimes R$$

for $X$ in $\mathcal{V}$.

We denote the category $\mathbf{D}^{b}_{\text{mot}}(\mathcal{V}) \otimes R$ by $\mathcal{D}$. We assume throughout this section that the conditions (4.3.1.2) are satisfied.
We have the functor (4.1.10.1)
\[ Z_{mot}(\ast): C^b_{mot}(\mathcal{V})^* \to C^{-}(Ab) \]
and the functor (4.2.18.1)
\[ Z^q_{/S}(\ast): L(\mathcal{V})^{op} \to C^{-}(Sh_{Zar}(\mathcal{V})) \]

Composing the functor \( Z^q_{/S}(\ast) \) with \( \mathcal{R}(\ast) \) gives the functor
\[ \mathcal{R}Z^q_{/S}(\ast): L(\mathcal{V})^{op} \to C^{-}(Ab)). \]

We now extend (4.3.2.1) to \( A_{mot}(\mathcal{V})^* \).

For an object of \( A^b_{mot}(\mathcal{V})^* \) of the form \( e a Z^X(b f[c]) \), we define
\[ \mathcal{R}Z_{mot}(e a Z^X(b f[c]), \ast) = \mathcal{R}Z^q_{X/S}(\ast)[c - 2q]; \]
we extend the definition of \( \mathcal{R}Z_{mot}(\Gamma, \ast) \) to arbitrary objects of \( A^b_{mot}(\mathcal{V})^* \) by taking direct sums.

To define \( \mathcal{R}Z_{mot}(q, \ast) \) for a morphism
\[ q: e^{\otimes a'} \otimes Z_Y(b')_g[c'] \to e^{\otimes a} \otimes Z_X(b)_f[c] \]
we use the representation of \( q \) as a sum of compositions of the form (4.2.4.1). If \( q \) is one such composition, say
\[ q = q(\tau, h_*, p), \]
we have the associated map of \( S \)-schemes (4.2.4.2)
\[ \bar{q} := q(\tau, h_*, p): X \to Y \]

For each open subscheme \( j: U \to Y \), we have the inclusion \( k: V \to X \), \( V = \bar{q}^{-1}(U) \), and the map (4.2.6.2)
\[ q_U: e^{\otimes a'} \otimes Z_U(b')_j \ast g[c'] \to e^{\otimes a} \otimes Z_V(b)_k \ast f[c] \]

Let
\[ Z_{mot}(1^{\otimes a'} \otimes Z_Y(b')_g[c'], \ast) \]
denote the complex of presheaves on \( Y \) defined by
\[ Z_{mot}(1^{\otimes a'} \otimes Z_X(b)_g[c'], \ast)(j: U \to Y) = Z_{mot}(1^{\otimes a'} \otimes Z_U(b')_j \ast g[c'], \ast), \]
and define the complex of presheaves on \( X \)
\[ Z_{mot}(e^{\otimes a} \otimes Z_X(b)_f[c], \ast) \]
similarly. The commutativity of the diagram (4.2.6.3) implies that the maps
\[
\mathcal{Z}_{mot}(q_U, *): \mathcal{Z}_{mot}(1^\otimes a' \otimes \mathbb{Z}_U(b_j^* g[c], *), \mathbb{Z}_V(b_k^* f[c], *) \to \mathcal{Z}_{mot}(e^\otimes a \otimes \mathbb{Z}_V(b_j^* f[c], *)
\]
define a map of complexes of presheaves
\[
\mathcal{Z}_{mot}(q, *): \mathcal{Z}_{mot}(1^\otimes a' \otimes \mathbb{Z}_X(b_j^* g[c], *), \mathbb{Z}_X(b_k^* f[c], *) \to \mathcal{Z}_{mot}(e^\otimes a \otimes \mathbb{Z}_X(b_k^* f[c], *)
\]
over the map \(\bar{q}\).

Taking the map of associated sheaves, and noting that
\[
\mathcal{Z}_{mot}(1^\otimes a \otimes \mathbb{Z}_W(q)[p], *) = \mathcal{Z}_{mot}(\mathbb{Z}_W(q)[h][p], *),
\]
we have the map of sheaves over \(q\):
\[
\mathfrak{F}(q, *): \mathfrak{F}_{Y/S}(q)[c' - 2b] \to \mathfrak{F}_{X/S}(f)[c - 2b]
\]
We let
\[
\mathfrak{R}_{3 mot}(q, *): \mathfrak{R}_{3 mot}(e^\otimes a' \otimes \mathbb{Z}_Y(b_j^* g[c], *), \mathbb{R}_{3 mot}(e^\otimes a \otimes \mathbb{Z}_X(b_k^* f[c], *) \to \mathfrak{R}_{3 mot}(e^\otimes a \otimes \mathbb{Z}_X(b_k^* f[c], *)
\]
be the map induced by \(\mathfrak{F}(q, *)\) on the global sections of the Godement resolution. We extend the definition of \(\mathfrak{R}_{3 mot}(q, *)\) to finite sums of compositions (4.2.4.1) by linearity, and extend to arbitrary maps between arbitrary objects of \(A^b_{mot}(V)^*\) by taking direct sums.

The relations (4.2.6.6)-(4.2.6.7) imply that the maps (4.3.2.2) define a DG functor
\[
\mathfrak{R}_{3 mot}(*): \mathfrak{A}_{mot}(V)^* \to \mathfrak{C}^-(\mathfrak{A}b).
\]

We may then extend (4.3.2.3) to the DG functor, compatible with cones
\[
\mathfrak{R}_{3 mot}(*): \mathfrak{C}^b_{mot}(V)^* \to \mathfrak{C}^-(\mathfrak{A}b))
\]
and the exact functor
\[
\mathfrak{R}_{3 mot}(*): \mathfrak{K}^b_{mot}(V)^* \to \mathfrak{K}^-(\mathfrak{A}b))
\]
by applying the functor \(\text{Tot} \circ \mathfrak{C}^b (\Pi, \S 2.1)\), and passing to the homotopy category.

We have the identity
\[
\mathfrak{R}_{3 mot}(e^\otimes a \otimes \mathbb{Z}_X(q)[2q], *) = \mathfrak{R}_{3 mot}^q_{X/S}(*)f,
\]
and the canonical isomorphism
\[
H^0(\mathfrak{R}_{3 mot}(e^\otimes a \otimes \mathbb{Z}_X(q)[2q - p], *)) \cong CH^q(X, p).
\]
More generally, for $\Gamma$ in $C^b_{mot}(\mathcal{V})^*$, we have the canonical isomorphism

$$H^0(\mathfrak{R}^\mathfrak{3}_{mot}(\Gamma, *)) \cong \mathcal{C}\mathcal{H}(\Gamma).$$

(4.3.2.5)

(4.3.3) Proposition

Under the assumption (4.3.1.2), the functor

$$\mathfrak{R}^\mathfrak{3}_{mot}(*) \otimes R: K^b_{mot}(\mathcal{V})^* \otimes R \rightarrow K^-(\text{Mod}_R)$$

extends to a functor of $R$-triangulated categories

$$\text{Re}^R_{\mathcal{C}\mathcal{H}}: \mathcal{D} \rightarrow \mathcal{D}^-(\text{Mod}_R),$$

Proof. The conditions (i) and (iii) of (4.3.1.2), together with the identity (4.3.2.4), imply that the functor $\mathfrak{R}^\mathfrak{3}_{mot}(*) \otimes R$ sends morphisms of (2.1.3)(a) and (d) to quasi-isomorphisms. The excision property (2.1.3)(b) is a general property of the functor $\mathfrak{R}$. The condition (ii) of (4.3.1.2) implies the morphisms of (2.1.3)(e) get sent to quasi-isomorphisms. For each connected $X$, the complex $\mathcal{Z}_{mot}(\mathcal{Z}X(0), *)$ is the complex

$$\ldots \rightarrow \mathcal{Z} \rightarrow \mathcal{Z} \rightarrow \ldots \rightarrow \mathcal{Z}$$

with the maps alternatively the identity map and the zero map; thus the canonical map

$$\mathcal{Z} \rightarrow \mathcal{Z}_{mot}(\mathcal{Z}X(0), *)$$

is a homotopy equivalence. From this, it is easy to verify that the morphism of (2.1.3)(f) gets sent to a quasi-isomorphism.

By (3.2.10), this implies that we have the desired extension.

\(\square\)

(4.3.4)

We denote the category $D^b_{mot}(\mathcal{V})^* \otimes R$ by $\mathcal{D}^*$, the category $K^b_{mot}(\mathcal{V})^* \otimes R$ by $\mathcal{K}^*$ and the category $C^b_{mot}(\mathcal{V})^* \otimes R$ by $\mathcal{C}^*$.

The identity

$$X = X \times \mathbb{Z}_S(0) = X \times \mathbb{Z}_{\Delta^0}(0)_{S^0}$$

for $X \in \mathcal{C}^*$ gives the natural transformation

$$\sigma_0: \mathcal{Z}_{mot}(-) \rightarrow \mathcal{Z}_{mot}(-, *).$$

(4.3.4.1)

Following $\sigma_0$ with the natural transformation (presheaf to associated sheaf to Godement resolution to complex of global sections):

$$\iota: \mathcal{Z}_{mot}(-, *) \rightarrow \mathfrak{R}^\mathfrak{3}_{mot}(-, *)$$

(4.3.4.2)
gives the natural transformation

\[ \mathcal{M}_0: \mathcal{Z}_{mot}(-) \to \mathcal{M}_{mot}(-, \ast). \] (4.3.4.3)

We denote the category \( K_B^{mot}(V)_B^* \) (see (3.2.3)) by \( K_B^* \). For \( \Delta \) in \( K_B^* \), we denote the map \( \mathcal{Z}_{mot}(e^{\otimes a} \otimes 1, \Delta) \) of (3.2.4)(iii) by

\[ \text{ev}^a_{\Delta}: \text{Hom}_{K_B^*}(e^{\otimes a} \otimes 1, \Delta) \to H^0(\mathcal{Z}_{mot}((\Delta))) \] (4.3.4.4)

If \( \Delta \) is in \( K_B^* \), then the map (4.3.4.4) is an isomorphism for all \( a \geq B \).

**Lemma (4.3.5)**

Let \( \Gamma \) and \( \Xi \) be objects of \( K_B^* \).

\[ f: \Gamma \to \Xi \]

a map in \( K^* \) which becomes an isomorphism in \( D^* \), and let

\[ g: e^{\otimes a} \otimes 1 \to \Xi \]

be a map in \( K^* \), with \( a \geq B \). Then there are hyper-resolutions (see (4.2.8))

\[ j_U: \Gamma \to \Gamma_U \]
\[ j_W: \Xi \to \Xi_W, \]

a map of hyper-resolutions over \( f \)

\[ \tilde{f}: \Gamma_U \to \Xi_W \]

and an integer \( N \) such that, for each \( n \geq N \), there is a map

\[ h_n: e^{\otimes a} \otimes 1 \to \Sigma^n \Gamma_U[n] \]

in \( K^* \) such that

\[ \Sigma^n(\tilde{f})[n] \circ h_n = i_n(\Xi_W) \circ j_W \circ g \]

in \( K^* \) (see (4.1.9) and (4.1.9.2) for the notation).

**Proof.** Since the map \( f \) becomes an isomorphism in \( D^* \), the map in \( D^-(\text{Mod}_R) \),

\[ \text{Re}_{\mathcal{CH}}^R(f): \text{Re}_{\mathcal{CH}}^R(\Gamma) \to \text{Re}_{\mathcal{CH}}^R(\Xi), \]

is an isomorphism. As

\[ \text{Re}_{\mathcal{CH}}^R(-) \circ \iota = \iota' \circ \mathcal{M}_{mot}(-, \ast) \otimes R \]

where

\[ \iota: K^* \to D^* \]
\[ \iota': K^-(\text{Mod}_R) \to D^-(\text{Mod}_R) \]
are the natural maps, there is an element \( \eta \) of \( H^0(\mathfrak{R}_{\text{mot}}(\Gamma, *)) \otimes R \) such that

\[
\mathfrak{R}_{\text{mot}}(f, *)(\eta) = \mathfrak{R}_{\text{mot}}(g)
\]  \hspace{1cm} (1)

in \( H^0(\mathfrak{R}_{\text{mot}}(\Xi, *) \otimes R) \) (see (4.3.4.3)).

As

\[
H^0(\mathfrak{R}_{\text{mot}}(\Gamma, *)) = \mathcal{CH}(\Gamma)
\]

\[
= \mathbb{H}^0_{\text{mot}}(\Gamma),
\]

(see (4.2.12.1), (4.3.2.5) and (4.2.13)) there is a hyper-resolution

\[ j_U : \Gamma \to \Gamma_U, \]

and an element \( \eta' \) of \( H^0(\mathcal{Z}_{\text{mot}}(\Gamma_U, *)) \) mapping to \( \eta \) under the natural map

\[ H^0(\mathcal{Z}_{\text{mot}}(\Gamma_U, *)) \to \mathbb{H}^0_{\text{mot}}(\Gamma). \]

By applying (4.2.10) repeatedly, we may assume that we have a hyper-resolution

\[ j_W : \Xi \to \Xi_W \]

of \( \Xi \), and a map of hyper-resolutions over \( f \)

\[ \tilde{f} : \Gamma_U \to \Xi_W. \]

By (1), the difference

\[ \mathcal{Z}_{\text{mot}}(\tilde{f}, *)(\eta') - \sigma_0(\text{ev}_a^{X_{iW}}(j_W \circ g)) \]

go to zero in \( \mathbb{H}_{\text{mot}}(\Xi) \); using (4.2.10) again, we may assume we have the identity

\[ \mathcal{Z}_{\text{mot}}(\tilde{f}, *)(\eta') = \sigma_0(\text{ev}_a^{X_{iW}}(j_W \circ g)) \]  \hspace{1cm} (2)

in \( H^0(\mathcal{Z}_{\text{mot}}(\Xi_W, *) \otimes R) \).

By (4.1.12), there is an integer \( N \) such that the natural maps (4.1.10.4)

\[
\Pi_n(\Gamma_U) : \mathcal{Z}_{\text{mot}}(\Sigma^n \Gamma_U[n]) \to \mathcal{Z}_{\text{mot}}(\Gamma_U, *)
\]

\[
\Pi_n(\Xi_W) : \mathcal{Z}_{\text{mot}}(\Sigma^n \Xi_W[n]) \to \mathcal{Z}_{\text{mot}}(\Xi_W, *)
\]

induces an isomorphism on \( H^0 \) for all \( n \geq N \). Take

\[ \eta_n \in H^0(\mathcal{Z}_{\text{mot}}(\Sigma^n \Gamma_U[n])) \otimes R \]

with

\[ \Pi_n(\eta_n) = \eta' \]
in $H^0(\mathcal{Z}_{\text{mot}}(\Gamma U, *))$. The relation (2) then gives the identity
\[
\mathcal{Z}_{\text{mot}}(\Sigma^n(\tilde{f})[n])(\eta_n) = \text{ev}_a^{\Sigma^n \Xi_W[n]}(i_n(\Xi_W) \circ j_W \circ g)
\] (3)
in $H^0(\mathcal{Z}_{\text{mot}}(\Sigma^n \Xi_W[n]) \otimes R)$.

By (3.2.4)(iii), there is a unique map
\[h_n: \mathfrak{e}^\otimes a \otimes 1 \to \Sigma^n \Gamma U[n]\]
in $\mathcal{K}^*$ such that
\[\text{ev}_a^{\Sigma^n \Gamma U[n]}(h_n) = \eta_n\]
in $H^0(\mathcal{Z}_{\text{mot}}(\Sigma^n \Gamma U[n]) \otimes R)$. The identity (3) implies the identity
\[\text{ev}_a^{\Sigma^n \Xi_W[n]}(\Sigma^n(\tilde{f})[n](h_n)) = \mathcal{Z}_{\text{mot}}(\Sigma^n(\tilde{f})[n])(\eta_n) = \text{ev}_a^{\Sigma^n \Xi_W[n]}(i_n(\Xi_W) \circ j_W \circ g)\]
in $H^0(\mathcal{Z}_{\text{mot}}(\Sigma^n \Xi_W[n]) \otimes R)$; applying (3.2.4) again, we have the identity of maps in $\mathcal{K}^*$
\[\Sigma^n(\tilde{f})[n](h_n)) = i_n(\Xi_W) \circ j_W \circ g,\]
completing the proof.

\[\square\]

(4.3.6)

We have the equivalence of triangulated categories ((3.2.5.1) and (3.2.6))
\[D^b_{\text{mot}}(r): D^b_{\text{mot}}(\mathcal{V}) \to D^b_{\text{mot}}(\mathcal{V})^*\]

For $\Gamma \in D^b_{\text{mot}}(\mathcal{V})$, we define
\[\mathcal{CH}(\Gamma) := \mathcal{CH}(D^b_{\text{mot}}(r)(\Gamma))\]
and define the map
\[\text{cl}(\Gamma): \mathcal{CH}(\Gamma) \to \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})}(1, \Gamma)\]
(4.3.6.1)
as the composition (see (4.2.16.1))
\[\mathcal{CH}(\Gamma) = \mathcal{CH}(\Gamma^*) \circ \text{cl}(\Gamma^*) \circ \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})^*}(1, \Gamma^*) \circ D^b_{\text{mot}}(r)(1, \Gamma)^{-1} \circ \text{Hom}_{D^b_{\text{mot}}(\mathcal{V})}(1, \Gamma)\]

where
\[\Gamma^* = D^b_{\text{mot}}(r)(\Gamma).\]
Theorem 4.3.7

Suppose the conditions (4.3.1.2) hold. Let $\Gamma$ be in $C^b_{mot}(V)$. Then the map (4.3.6.1)

$$\text{cl}(\Gamma): CH(\Gamma) \otimes R \to \text{Hom}_{D^b_{mot}(V)}(1, \Gamma) \otimes R$$

is surjective. In particular, the map

$$\text{cl}^q_p: CH^q(X, p) \otimes R \to H^{q-p}(X, R(q))$$

is surjective for all $X$ in $V$.

Proof. Using the equivalence $D^b_{mot}(r)$, we may replace $D^b_{mot}(V)$ with $D^*$, and assume that $\Gamma$ is in $D^*$.

We have the isomorphism (2.2.4.1) in $D^*$:

$$\nu_1: \epsilon \otimes 1 \to 1.$$  

Since $D^*$ is a localization of $K^*$, each map

$$\phi: 1 \to \Gamma$$

in $D^*$ may be factored as a composition

$$1 \xrightarrow{(\nu_1)^{-1}} \epsilon \otimes 1 \xrightarrow{g} \Xi \xrightarrow{f^{-1}} \Gamma,$$

where $g: \epsilon \otimes 1 \to \Xi$ and $f: \Gamma \to \Xi$ are maps in $K^*$, and $f$ is invertible in $D^*$.

Applying (4.3.5), there are hyper-resolutions

$$j_U: \Gamma \to \Gamma_U$$

$$j_W: \Xi \to \Xi_W,$$

an integer $n$, and maps

$$h_n: \epsilon \otimes 1 \to \Sigma^n \Gamma_U[n]$$

$$\tilde{f}: \Gamma_W \to \Xi_U$$

in $K^*$ such that

$$\Sigma^n(\tilde{f})[n] \circ h_n = i_n(\Xi_W) \circ j_W \circ g. \quad (1)$$

In addition, the diagram

$$\begin{array}{ccc}
\Gamma & \xrightarrow{f} & \Xi \\
\downarrow{j_U} & & \downarrow{j_W} \\
\Gamma_U & \xrightarrow{\tilde{f}} & \Xi_W \\
\downarrow{i_n(\Gamma_U)} & & \downarrow{i_n(\Xi_W)} \\
\Sigma^n \Gamma_U[n] & \xrightarrow{\Sigma^n(\tilde{f})[n]} & \Sigma^n \Xi_W[n]
\end{array} \quad (2)$$
commutes in $K^*$. Since $i_n(\Gamma_U)$, $j_U$, $i_n(\Xi_W)$ and $j_W$ are isomorphisms in $D^*$, the relation (1) and the commutativity of (2) gives us the identity

$$f^{-1} \circ g \circ (\nu_1)^{-1} = f^{-1} \circ (j_W)^{-1} \circ (i_n(\Xi_W))^{-1} \circ \Sigma^n(f)[n] \circ h_n \circ (\nu_1)^{-1}$$

$$= (j_U)^{-1} \circ (i_n(\Gamma_U))^{-1} \circ h_n \circ (\nu_1)^{-1}$$

(3)

Let $\tilde{\eta}$ be the image of $\text{ev}_{\Sigma^n \Gamma_U[n]}(h_n)$ (4.3.4.4) in $H^0(\mathcal{Z}_{mot}(\Sigma^n \Gamma_U[n], *)) \otimes R$, under the map (4.3.4.1)

$$\sigma_0(\Sigma^n \Gamma_U[n]): \mathcal{Z}_{mot}(\Sigma^n \Gamma_U[n]) \otimes R \to \mathcal{Z}_{mot}(\Sigma^n \Gamma_U[n], *) \otimes R.$$  

By (4.2.13), we have

$$\mathcal{C}H(\Gamma) = \mathbb{H}^0_{\mathcal{Z}_{mot}(*)}(\Gamma),$$

hence $\tilde{\eta}$ has a well-defined image $\eta \in \mathcal{C}H(\Gamma) \otimes R$. By definition of the map (4.2.16.1),

$$\text{cl}(\Gamma): \mathcal{C}H(\Gamma) \to \text{Hom}_{D_{mot}(\mathcal{V})}(1, \Gamma),$$

we have

$$\text{cl}(\Gamma)(\eta) = (j_U)^{-1} \circ (i_n(\Gamma_U))^{-1} \circ h_n \circ (\nu_1)^{-1};$$

as this is the map $\phi$ by (3), surjectivity is proved. \hfill \Box

4.4. Injectivity of the cycle map

We give a general criterion for the injectivity of the cycle map.

4.4.1 Cohomology vanishing

In order to prove injectivity, we need, in addition to the hypotheses (4.3.1.2), the following additional hypothesis: Let

$$p: X \times_S \mathbb{A}^n \to X$$

be the projection, and let $(\mathbb{A}^n_S, g)$ and $(X, f)$ be liftings of $\mathbb{A}^n$ and $X$ to objects of $\mathcal{L}(\mathcal{V})$. Then the map

$$p^*: \mathcal{Z}_X^0(*)_f \to p_*(\mathcal{Z}_{X \times_S \mathbb{A}^n_S}^q(*)_f \times g)$$

is a quasi-isomorphism. \hfill \Box

4.4.2 A double cycle complex

Recall from §4.1 that the cosimplicial scheme

$$\Delta^*: \Delta \to \mathcal{V}$$
Injectivity of the cycle map

(4.1.8) \((\Delta^*, \delta^*): \Delta \to \mathcal{L}(\mathcal{V})\),

and the associated simplicial object \(\mathbb{Z}_{\Delta^*}(0)\delta^*\) (4.1.9)(iii) of \(\mathcal{A}_{mot}(\mathcal{V})^*\). For \(\Gamma \in \mathcal{A}_{mot}(\mathcal{V})^*\), the complex \(\mathbb{Z}_{mot}(\Gamma, \ast)\) is the complex associated to the simplicial object

\[ \mathbb{Z}_{mot}(\Gamma \times \mathbb{Z}_{\Delta^*}(0)\delta^*): \Delta^{op} \to \mathbb{C}^b(\mathbb{A}b) \]

(see (4.1.10)). We now form the bi-simplicial object

\[ \mathbb{Z}_{mot}(\Gamma \times \mathbb{Z}_{\Delta^*}(0)\delta^* \times \mathbb{Z}_{\Delta^*}(0)\delta^*): \Delta^{op} \times \Delta^{op} \to \mathbb{C}^b(\mathbb{A}b). \]

and let \(\mathbb{Z}_{mot}(\Gamma, \ast, \ast)\) be the associated double complex. Since \((\mathbb{Z}_{\Delta^*}(0)\delta^*)_0 = \mathbb{Z}_S(0)\), and \(\mathbb{Z}_S(0)\) is the unit for the tensor operation \(\times\), the sub-complexes \(\mathbb{Z}_{mot}(\Gamma, \ast, 0)\) and \(\mathbb{Z}_{mot}(\Gamma, 0, \ast)\) are canonically isomorphic to the complex \(\mathbb{Z}_{mot}(\Gamma, \ast)\). This defines the two inclusions

\[ i_1, i_2: \mathbb{Z}_{mot}(\Gamma, \ast) \to \text{Tot}(\mathbb{Z}_{mot}(\Gamma, \ast, \ast)). \tag{4.4.2.1} \]

We let \(\mathcal{Z}^q(X, \ast)_f\) denote the complex \(\mathbb{Z}_{mot}(\mathcal{Z}_X(q)_f[2q], \ast)\), and we define the double complex \(\mathcal{Z}^q(X, \ast, \ast)_f\) by

\[ \mathcal{Z}^q(X, \ast, \ast)_f = \mathbb{Z}_{mot}(\mathcal{Z}_X(q)[2q], \ast, \ast). \tag{4.4.2.2} \]

the inclusions (4.4.2.1) give the natural maps

\[ i_1, i_2: \mathcal{Z}^q(X, \ast)_f \to \text{Tot}(\mathcal{Z}^q(X, \ast, \ast)_f). \tag{4.4.2.3} \]

We may sheafify this construction over \(X\); let \(\mathcal{Z}^q_{X/S}(\ast, \ast)_f\) be the double complex of sheaves on \(X\) associated to the presheaf

\[(j: U \to X) \mapsto \mathcal{Z}^q(U, \ast, \ast)_j; \]

the maps (4.4.2.3) define the maps

\[ i_1, i_2: \mathcal{Z}^q_{X/S}(\ast) \otimes R \to \text{Tot}(\mathcal{Z}^q_{X/S}(\ast, \ast)_f) \otimes R. \tag{4.4.2.4} \]

(4.4.3) Lemma

Assume that the hypothesis (4.4.1) holds. Then the maps (4.4.2.4) are quasi-isomorphisms.

Proof. We consider one of the two convergent spectral sequences associated to the double complex of sheaves \(\mathcal{Z}^q_{X/S}(\ast, \ast)_f\). The \(E_1\)-terms are given by

\[ E_1^{a, b} = \mathcal{H}^a(\mathcal{Z}^q_{X/S}(\ast, b)_f) \otimes R \]
where \( 3^q_{X/S}(*, b)_f \) is the complex of sheaves on \( X \) associated to the presheaf

\[
U \mapsto Z^q(X \times \Delta^b, *)_{f \times \delta^b}.
\]

and \( H^a \) is the sheaf of cohomology groups on \( X \).

Let

\[
p^n: X \times_S \Delta^n \to X
\]

be the projection. We have the identity

\[
3^q_{X/S}(*, b)_f = p^b_*(3^q_{X \times S \Delta^b/S}(*, f \times \delta^b)).
\]

By our assumption (4.4.1), the map

\[
p^{b*}: 3^q_{X/S}(*)_f \otimes R \to 3^q_{X/S}(*, b)_f \otimes R
\]

is a quasi-isomorphism. This implies that the complex of \( E_1 \)-terms is

\[
\cdots \to H^a(3^q_{X/S}(*)_f) \otimes R \to H^a(3^q_{X/S}(*)_f) \otimes R \to \cdots \to H^a(3^q_{X/S}(*)_f) \otimes R
\]

where the maps alternate between the zero map and the identity map, with the last map being the zero map. Thus, the spectral sequence degenerates at \( E_2 \), and the inclusion

\[
i_1: 3^q_{X/S}(*)_f = 3^q_{X/S}(*, 0)_f \otimes R \to \text{Tot}(3^q_{X/S}(*, *)) \otimes R
\]

is a quasi-isomorphism. The other inclusion \( i_2 \) is handled by using the other spectral sequence.

\[\square\]

(4.4.4)

We now return to the cosimplicial object

\[(X, f) \times (\Delta^*, \delta^*): \Delta \to \mathcal{L}(\mathcal{V}).\]

We may view the double complex \( Z^q(X, *, *)_f \) as the double complex associated to the simplicial object

\[
Z^q(X \times_S \Delta^*, *)_{f \times \delta^*}: \Delta^{\text{op}} \to \mathbb{C}^\text{Ab}
\]

\[
n \mapsto Z^q_{\text{mot}}(X \times_S \Delta^n, *)_{f \times \delta^n}.
\]

We may apply the natural transformation (4.3.4.2)

\[
i_{Y, g}: Z^q(Y, *)_g \to \mathfrak{M}^q_{Y/S}(*)_g
\]
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giving the simplicial object

\[ \mathcal{R}^q_{X \times S^* S}(* \times \delta^* : \Delta^op \to C^- \text{Ab} \]

\[ n \mapsto \mathcal{R}^q_{X \times S^* S}(*) \times \delta^n, \]

and the natural map of simplicial complexes

\[ \iota_{X \times S^* S} : Z^q(X \times S^* S, *) \times \delta* \to \mathcal{R}^q_{X \times S^* S}(*) \times \delta^n. \]

We let

\[ \iota_{X}(*)_{f} : Z^q(X, *, *) \to \mathcal{R}^q_{X/S}(*)f. \tag{4.4.4.1} \]

denote the induced map on the associated double complexes; here the indices in the double complex \( \mathcal{R}^q_{X/S}(*)f \) are arranged so that

\[ \mathcal{R}^q_{X/S}(*)f(n) = \mathcal{R}^q_{X \times S^* S}(*) \times \delta^n. \]

One easily sees that the map (4.4.4.1) factors canonically through the natural map

\[ Z^q(X, *, *) \to \mathcal{R}(Z^q_{X/S}(*, *)) \]

giving the map

\[ \mathcal{R}^q_{\iota X} : \mathcal{R}(Z^q_{X/S}(*, *)) \otimes R \to \mathcal{R}^q_{X/S}(*) \times \delta^n. \tag{4.4.4.2} \]

(4.4.5) Lemma

Assume the conditions (4.3.1.2) and (4.4.1) hold. Then the map (4.4.4.2) induces a quasi-isomorphism on the associated total complexes.

Proof. By (4.4.1) and (4.3.1.2)(ii), the map

\[ p^n : \mathcal{R}^q_{X/S}(*) \to \mathcal{R}^q_{X \times S^* S}(*) \times \delta^n \tag{1} \]

is a quasi-isomorphism for each \( n \). The same spectral sequence argument as in the proof of (4.4.3) shows that the inclusion

\[ \iota_1 : \mathcal{R}^q_{X/S}(*)f = \mathcal{R}^q_{X/S}(0)_f \to \text{Tot}(\mathcal{R}^q_{X/S}(*)_f) \]

is a quasi-isomorphism. By (4.4.3), the inclusion

\[ \mathcal{R}(i_1) : \mathcal{R}^q_{X/S}(*)f = \mathcal{R}(Z^q_{X/S}(0)_f) \to \text{Tot}(\mathcal{R}^q_{X/S}(*, *)_f) \]

is a quasi-isomorphism. As

\[ \mathcal{R}^q_{\iota X} \circ \mathcal{R}(i_1) = \iota_1, \]

the lemma is proved.
(4.4.6)

Let $\Gamma$ be in $C_{mot}^b(V)^\ast$. We may form the functor

$$Z^{\Gamma}_{mot}: \mathcal{L}(V)^{op} \to C^b(Ab)$$

defined by

$$Z^{\Gamma}_{mot}(-) = Z_{mot}(\Gamma \times (-)).$$

For $(X, f) \in \mathcal{L}(V)$, we may form the presheaf on $X$:

$$(j: U \to X) \mapsto Z^{\Gamma}_{mot}(\mathbb{Z}_U(q) j^* f[2q]);$$

we let $\mathcal{Z}_{X/S, f}^q$ denote the associated sheaf. The natural transformation (4.3.4.3)

$$\mathfrak{RZ}_0: Z_{mot}(-) \to \mathfrak{RZ}_{mot}(-, \ast)$$

defines the natural map

$$\phi_{\Gamma}: \mathfrak{Z}_{X/S, f}^{q, \Gamma} \to \mathfrak{RZ}_{mot}(\Gamma \times \mathbb{Z}_X(q)f[2q], \ast) \tag{4.4.6.1}$$

We have the object (4.1.8.2) $\mathbb{Z}_{\Delta^N}^{\leq 0}$ of $C_{mot}^b(V)^\ast$, the functor $\Sigma^N[N] = \mathbb{Z}_{\Delta^N}^{\leq 0} \times (-)$ (4.1.9), and the functor (4.2.12.3)

$$\Sigma^N Z_{mot}[N] = Z_{mot} \circ \Sigma^N(-)[N]: C_{mot}^b(V)^\ast \to C^b(Ab),$$

Using the above notation, we may write $\Sigma^N Z_{mot}[N]$ as

$$\Sigma^N Z_{mot}[N] = Z_{mot}^{\mathbb{Z}_{\Delta^N}^{\leq 0}}(-)$$

Denote $\Sigma^N Z_{mot}[N](\mathbb{Z}_X(q)f[2q])$ by $\Sigma^N Z_{mot}^q(X)f[N]$. As above, we may sheafify the Zariski presheaf

$$j: U \to X \mapsto \Sigma^N Z_{mot}^q(U) j^* f[N]$$

over $X$, giving the complex of sheaves $\Sigma^N \mathfrak{Z}_{X/S, f}^q[N]$ on $X$, and the functor

$$\Sigma^N \mathfrak{Z}_{X/S, -}^q[N]: \mathcal{L}(V)^{op} \to C^-(Sh_{Zar, S}(Ab))$$

We have the identity

$$\Sigma^N \mathfrak{Z}_{X/S, -}^q[N] = \mathfrak{Z}_{X/S, f}^{q, \Delta^N}$$

giving the natural map (4.4.6.1)

$$\phi_{X,N}: \mathfrak{R} \Sigma^N \mathfrak{Z}_{X/S, -}^q[N] \otimes R \to \mathfrak{RZ}_{mot}(\mathbb{Z}_{\Delta^N}^{\leq 0} \times \mathbb{Z}_X(q)f[2q], \ast) \otimes R \tag{4.4.6.2}$$
The natural transformation (4.2.12.4) gives the natural transformations

\[ \mathfrak{Z} \Pi_{N}: \Sigma^{N} \mathfrak{Z}_{S}^{q} \rightarrow \mathfrak{Z}_{S}^{q}(\text{--}) \]

\[ \mathfrak{R} \mathfrak{Z} \Pi_{N}: \mathfrak{R} \Sigma^{N} \mathfrak{Z}_{S}^{q} \rightarrow \mathfrak{R} \mathfrak{Z}_{S}^{q}(\text{--}) \]

We recall from (4.1.7) and (4.1.8) that \( Z_{\Delta}^{\leq N}(0) \) is the complex

\[ Z_{\Delta}^{\leq N}(0)^{-N} \rightarrow \cdots \rightarrow Z_{\Delta}^{\leq N}(0)^{0}, \]

where \( Z_{\Delta}^{\leq N}(0)^{-p} \) is the direct sum

\[ Z_{\Delta}^{\leq N}(0)^{-p} = \oplus \{ g: [p] \rightarrow [N] Z_{\Delta^p}(0) \delta^p \}, \]

where the sum is over injective ordered maps \( g \). The map \( \Pi_{N} \) in each degree \( p \) is the map induced on \( Z_{mot}(\text{--}) \) by the sum map

\[ \Sigma_{N,p}: \oplus \{ g: [p] \rightarrow [N] Z_{\Delta^p}(0) \delta^p \} \rightarrow Z_{\Delta^p}(0) \delta^p. \]

Applying the functor \( \mathfrak{R} \mathfrak{Z}_{S}(-, \text{--}) \) to \( \Sigma_{N,p} \times \text{id} \) gives the natural map

\[ \mathfrak{R} \mathfrak{Z} \Pi_{N}(\text{--}): \mathfrak{R} Z_{mot}(Z_{\Delta}^{\leq N}(0) \times Z_{X}(q)[2q], \text{--}) \rightarrow \mathfrak{R} Z_{X/S}(\text{--})(\text{--}) \]

(4.4.7) **Lemma**

Let \( p \) be an integer. For fixed \( X, f \) and \( q \), there is an integer \( N_{p} \) such that the map (4.4.6.2) induces an isomorphism in cohomology \( H^{m}(\text{--}) \otimes R \) for all \( m \geq -p \) if \( N \geq N_{p} \).

**Proof.** By (4.1.12) and (4.2.14), there is an \( N_{p} \) such that the map (4.4.6.3)

\[ \mathfrak{R} \Pi_{N}: \mathfrak{R} \Sigma^{N} \mathfrak{Z}_{X,S,f}^{0} \rightarrow \mathfrak{R} \mathfrak{Z}_{X/S}^{q}(\text{--}) \]

is an isomorphism on \( H^{m} \) for \( N \geq N_{p} \) and \( m \geq -p \).

We have the natural transformation (4.1.9.2)

\[ i_{N}: \text{id} \rightarrow \Sigma^{N} [N], \]

inducing the natural map

\[ \mathfrak{R} Z_{mot}(i_{N}(Z_{X}(q)[2q])): \mathfrak{R} Z_{mot}(Z_{X}(q)[2q], \text{--}) \otimes R \rightarrow \mathfrak{R} Z_{mot}(Z_{\Delta}^{\leq N}(0) \times Z_{X}(q)[2q], \text{--}) \otimes R \]

(2)
By (4.1.15), the map
\[ i_N(\mathbb{Z}_X(q)_f[2q]) : \mathbb{Z}_X(q)_f[2q] \to \mathbb{Z}_X^{\leq N}(0) \times \mathbb{Z}_X(q)_f[2q] \]
is an isomorphism in $D^b_{mot}(\mathcal{V})$. By (4.3.3), the map (2) is thus a quasi-isomorphism. We have the identity
\[ \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) = \mathcal{R}^q_{X/S}(*) \cdot f. \]
Let
\[ i_{X,j} : \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \to \mathcal{R}^q_{X/S}(*) \cdot f; \quad j = 1, 2, \]
be the composition
\[ \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \xrightarrow{\mathcal{R}^q_{mot}(i_{X,j})} \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \xrightarrow{\mathcal{R}^q_{mot}(\cdot f) \cdot f} \mathcal{R}^q_{X/S}(*) \cdot f. \]
We have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) & = & \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \\
\downarrow \mathcal{R}^q_{mot}(i_{N}(\mathbb{Z}_X(q)_f[2q])) & & \downarrow i_{X,1} \\
\mathcal{R}^q_{mot}(\mathbb{Z}_X^{\leq N}(0) \times \mathbb{Z}_X(q)_f[2q], *) & \xrightarrow{\mathcal{R}^q_{mot}(\cdot f) \cdot f} & \mathcal{R}^q_{X/S}(*) \cdot f \\
\end{array}
\]
By (4.4.3) and (4.4.5), the map
\[ i_{X,1} : \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \otimes R \to \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \otimes R \]
is a quasi-isomorphism. As the map (2) is a quasi-isomorphism, the map (4.4.6.4) induces a quasi-isomorphism
\[ \mathcal{R}^q_{mot}(\mathbb{Z}_X^{\leq N}(0) \times \mathbb{Z}_X(q)_f[2q], *) \otimes R \to \mathcal{R}^q_{mot}(\mathbb{Z}_X^{\leq N}(0) \times \mathbb{Z}_X(q)_f[2q], *) \otimes R. \tag{3} \]
By (4.4.3) and (4.4.5), the map
\[ i_{X,2} : \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \otimes R \to \mathcal{R}^q_{mot}(\mathbb{Z}_X(q)_f[2q], *) \otimes R \]
is a quasi-isomorphism. One easily sees that the diagram
\[
\begin{array}{ccc}
\mathcal{R}^q_{X/S, *} \otimes [N] & \xrightarrow{\phi_X, N} & \mathcal{R}^q_{mot}(\mathbb{Z}_X^{\leq N}(0) \times \mathbb{Z}_X(q)_f[2q], *) \\
\mathcal{R}^q_{X/S, *} \otimes [N] & \xrightarrow{i_{X,2}} & \mathcal{R}^q_{X/S}(*) \otimes [N] \\
\end{array}
\]
commutes. This, together with (1), (3) and (4), proves the lemma.\[ \square \]
Suppose the conditions (4.3.1.2) and (4.4.1) are satisfied. Then the map
\[ \text{cl}(\Gamma): \text{CH}(\Gamma) \otimes R \to \text{Hom}_{\mathcal{DM}(\mathcal{V})}(1, \Gamma) \otimes R \]
is an isomorphism for all \( \Gamma \) in \( \mathcal{DM}(\mathcal{V}) \).

Proof. It suffices to prove the result for \( \Gamma \) in \( \mathbf{D}^b_{\text{mot}}(\mathcal{V}) \); using the equivalence ((3.2.5.1) and (3.2.6))

\[ \mathbf{D}^b_{\text{mot}}(r): \mathbf{D}^b_{\text{mot}}(\mathcal{V}) \to \mathbf{D}^b_{\text{mot}}(\mathcal{V})^*, \]
we may assume \( \Gamma \) is in \( \mathbf{D}^b_{\text{mot}}(\mathcal{V})^* \). As \( \mathbf{D}^b_{\text{mot}}(\mathcal{V})^* \) is generated as a triangulated category by the objects \( \epsilon^{\otimes a} \otimes Z_X(q)_f \), and since \( \text{cl}(-) \) is an exact natural transformation of cohomological functors we may take \( \Gamma \) to be a translate of \( \epsilon^{\otimes a} \otimes Z_X(q)_f \); as \( \epsilon^{\otimes a} \otimes Z_X(q)_f \) is isomorphic to \( Z_X(q)_f \) in \( \mathbf{D}^b_{\text{mot}}(\mathcal{V}) \), we may take \( \Gamma \) to be a translate of \( Z_X(q)_f \). By (4.3.7), we need only prove injectivity.

By (4.2.15), there is an \( N_1 \) such that
\[ \text{CH}(Z_X(q)_f[2q - p]) \otimes R \cong H^0_{\Sigma^N Z_{\text{mot}}[N]}(Z_X(q)_f[2q - p]) \otimes R \] (1)
for all \( N \geq N_1 \), with the isomorphism defined via the map \( H^0(\Pi_N) \). We may identify the hypercohomology with respect to \( \Sigma^N Z_{\text{mot}}[N] \) as Zariski hypercohomology:
\[ H^0_{\Sigma^N Z_{\text{mot}}[N]}(Z_X(q)_f[2q - p]) \otimes R = H^{-p}(\mathcal{R}^{\Sigma^N \mathcal{Z}_{X/S,f}[N]} \otimes R. \] (2)

It follows directly from the construction of \( \text{Re}^R_{\text{CH}} \) in (4.3.3) that the composition
\[ \text{Re}^R_{\text{CH}} \circ \text{cl}(Z_X(q)_f[2q - p]): \text{CH}(Z_X(q)_f[2q - p]) \otimes R \to H^0(\text{Re}^R_{\text{CH}}(Z_X(q)_f[2q - p])) \] (3)
is the map induced on \( H^{-p} \) by the map (4.6.2)
\[ \phi_{X,N}: \mathcal{R}^{\Sigma^N \mathcal{Z}_{X/S,f}[N]} \to \mathcal{R}_{\text{mot}}(Z_{\Delta}^N(0) \times Z_X(q)_f[2q], *) \]
once we identify \( \text{CH}(Z_X(q)_f[2q - p]) \otimes R \) with \( H^{-p}(\mathcal{R}^{\Sigma^N \mathcal{Z}_{X/S,f}[N]} \otimes R \) via (1) and (2). By (4.4.7), the map (3) is an isomorphism, once we take \( N \) large enough. Thus \( \text{cl}(Z_X(q)_f[2q - p]) \) is injective, completing the proof. \( \square \)

We recall from (3.2.11) the triangulated tensor category \( \mathcal{DM}(\mathcal{V})^0 \), and the exact tensor functor
\[ \mathcal{DM}(H_{\text{mot}}): \mathcal{DM}(\mathcal{V}) \to \mathcal{DM}(\mathcal{V})^0. \]
(4.4.9) Theorem

Suppose the conditions (4.3.1.2) and (4.4.1) are satisfied. Then the functor $\mathcal{D}M(H_{mot})$ induces an isomorphism

$$\mathcal{D}M(H_{mot}): \text{Hom}_{\mathcal{D}M(V)}(1, \Gamma) \otimes R \to \text{Hom}_{\mathcal{D}M(V)^0}(1, \mathcal{D}M(H_{mot})(\Gamma)) \otimes R$$

for all $\Gamma$ in $\mathcal{D}M(V)$.

Proof. We refer throughout to the notations of (3.2.11). The arguments of §4.1-§4.4 can be applied, replacing the categories $A_{mot}(V)^{0*}$, $C^{b}_{mot}(V)^{0*}$, $K^{b}_{mot}(V)^{0*}$, etc., with $A^{0*}_{mot}(V)$, $C^{b}_{mot}(V)^{0*}$, $K^{b}_{mot}(V)^{0*}$, etc. verifies the analog of (4.4.8) for the category $\mathcal{D}M(V)^0$, i.e., that there is a natural cycle class map

$$\text{cl}(\Gamma): \text{CH}(\Gamma) \otimes R \to \text{Hom}_{\mathcal{D}M(V)^0}(1, \Gamma) \otimes R,$$

which is an isomorphism for all $\Gamma$ in $\mathcal{D}M(V)^0$.

Noting that the construction of $\text{cl}(\cdot)$ is compatible with the functor $H_{mot}$ proves the results. $\square$

4.5. Some moving lemmas

In this section, we verify a version of the classical moving lemma for the complexes $Z^q(X, *)_f$ in case the base $S$ is $\text{Spec}(k)$ for a field $k$, and $X$ is affine. This helps in the next section, where we verify the criteria (4.3.1.2) and (4.4.1) in case $S$ is smooth, essentially of finite type, and of dimension $\leq 1$ over a field $k$.

The main results of this section have been also proved by Bloch [B3] by essentially the same method; as this work has not appeared in published form, we give the details here. We take $S = \text{Spec}(k)$ throughout this section.

(4.5.1)

We call a subvariety of $\Delta^p$ of the form $\Delta^*(h)(\Delta^m)$ for some $h: [m] \to [p]$ in $\Delta$ a face of $\Delta^p$; all faces $F$ of $\Delta^p$ are given by equations of the form

$$t_{i_1} = \ldots = t_{i_s} = 0$$

where

$$\Delta^p = \text{Spec}(k[t_0, \ldots, t_p]/\sum_{i=0}^{p} t_i - 1).$$

Let $p_X: X \to S$ be a smooth $k$-variety, and $\mathcal{C} = \{C_1, \ldots, C_s\}$ a finite collection of irreducible locally closed subsets of $X$; let

$$i_j: C_j \to X$$
be the inclusion. Let $m = (m_1, \ldots, m_s)$ be a sequence of integers such that $m_j \leq q$, $j = 1, \ldots, s$, and let $\mathcal{Z}_{C,m}^q(X,p)$ be the subgroup of $\mathcal{Z}(X,p)$ generated by the codimension $q$ subvarieties $W$ of $X \times \Delta^p$ such that, for each face $F$ of $\Delta^p$ and each $i$, we have

$$\text{codim}_{C_i \times F}(W \cap (C_i \times F)) \geq m_i$$

or the intersection is empty.

(4.5.2) Lemma

Let $(X,f)$ be in $\mathcal{L}(\text{Sm}_k)$. Then the complex $\mathcal{Z}(X,*)_f$ is equal to $\mathcal{Z}_{C,m}^q(X,*)$ for some finite set of locally closed irreducible subsets $C$, and some sequence $m$.

Proof. Write $f$ as

$$f: X' \to X.$$ 

Write $X'$ as a union of connected components

$$X' = \prod_{i=1}^{s} X'_i,$$

and let

$$f_i: X'_i \to X$$

be the restriction of $f$ to $X'_i$. As $X'$ is smooth over $k$, each $X'_i$ is irreducible; let $n_i = \dim_k(X'_i)$.

Let $C_{i,j}$ be the subset of $X$ defined by

$$x \in C_{i,j} \iff \dim_k(f_i^{-1}(x)) = j.$$ 

The sets $C_{i,j}$ are locally closed subsets of $X$, and form a filtration of the constructible subset $f_i(X'_i)$ of $X$. Write each $C_{i,j}$ as a union of irreducible components

$$C_{i,j} = \cup_l C^l_{i,j}$$

and let

$$d^l_{i,j} = \dim_k(C^l_{i,j}).$$

Clearly, we have

$$d^l_{i,j} + j \leq n_i. \quad (1)$$

Now let $W$ be a reduced irreducible codimension $q$ closed subset of $X \times \Delta^p$, $g: \Delta^m \to \Delta^p$ a face of $\Delta^p$, and let $F = g(\Delta^m)$. Let $W'$ be an irreducible component of $(f_i \times g)^{-1}(W)$; then there is an $j$ and an irreducible component $C^l_{i,j}$ of $C_{i,j}$ such that

$$(f_i \times g)(W') \subset C^l_{i,j} \times F$$

and

$$(f_i \times g)(W') \not\subset C_{i,j+1} \times F.$$
From this it follows that
\[ \dim_k(W') = j + \dim_k((f_i \times g)(W')) \leq j + \dim_k(W \cap (C_{i,j}^l \times F)). \] (2)

Now suppose that
\[ \operatorname{codim}_{X' \times \Delta^m}(W') < q. \]
Then (2) implies
\[ n_i + m - q < \dim_k(W') \]
\[ \leq j + \dim_k(W \cap (C_{i,j}^l \times F)) \]
\[ = j + d_{i,j}^l + m - \operatorname{codim}_{C_{i,j}^l \times F}(W \cap (C_{i,j}^l \times F)) \]
or
\[ \operatorname{codim}_{C_{i,j}^l \times F}(W \cap (C_{i,j}^l \times F)) < j + d_{i,j}^l - n_i + q. \] (3)

Conversely, suppose that (3) holds for some \( i, j, l \). Take an irreducible component \( Z \) of the intersection \( W \cap (C_{i,j}^l \times F) \) of maximal dimension; then
\[ \dim_k((f_i \times g)^{-1}(W)) \geq \dim_k((f_i \times g)^{-1}(Z)) \]
\[ \geq j + \dim_k(Z) \]
\[ > j + d_{i,j}^l + m - (j + d_{i,j}^l - n_i + q) \]
\[ = n_i + m - q. \]

Thus, if we let \( m_{i,j}^l \) be defined by
\[ m_{i,j}^l = j + d_{i,j}^l - n_i + q \]
then
\[ \operatorname{codim}_{X' \times \Delta^m}((f \times g)^{-1}(W)) \geq q \quad \text{for all } g: \Delta^m \to \Delta^p \]
\[ \iff \operatorname{codim}_{C_{i,j}^l \times F}(W \cap (C_{i,j}^l \times F)) \geq m_{i,j}^l \quad \text{for all } i, j, l \text{ and all faces } F. \]

In addition, by (1), we have
\[ m_{i,j}^l \leq q \]
This gives the equality
\[ Z^q(X_i, *)_f = Z^q_{C, m}(X, *) \]
for
\[ C = \{ \ldots, C_{i,j}, \ldots \}; \quad m = (\ldots, m_{i,j}, \ldots) \]
(4.5.3) Generic projections

Let $X$ be a smooth affine $k$-variety of dimension $n$, embedded as a closed subset of $\mathbb{A}^N$, with $N > n$. We let $\bar{X}$ be the closure of $X$ in $\mathbb{P}^N \supset \mathbb{A}^N$. Let $\mathbb{P}^{N-1} \supset \mathbb{A}^N$, and $\bar{X}_\infty$ the intersection $\bar{X} \cap \mathbb{P}^{N-1}_\infty$.

For a linear subvariety $L \subset \mathbb{P}^N$ of dimension $N - n - 1$, we let

$$\pi_L: \mathbb{P}^N - L \to \mathbb{P}^n$$

denote the projection with center $L$; the projection with center $L \subset \mathbb{P}^{N-1}_\infty$ gives the affine-linear map

$$\pi^0_L: \mathbb{A}^N \to \mathbb{A}^n.$$

The restriction of $\pi^0_L$ to $X$:

$$\pi_{L,X}: X \to \mathbb{A}^n$$

is finite if and only if $L \cap \bar{X} = \emptyset$. We let $U_X$ denote the subset of the Grassmannian $\text{Gr}_{\mathbb{P}^{N-1}}(n + 1) := \text{Gr}(N - 1, n + 1)$ of $L$ with $L \cap \bar{X} = \emptyset$.

For a collection of locally closed irreducible subsets

$$C = \{C_1, \ldots, C_s\}$$

and sequence of integers $m = (m_1, \ldots, m_s)$ with $m_i \leq q$, we let $m - 1$ be the sequence $(m_1 - 1, \ldots, m_s - 1)$ and $m + 1$ the sequence $(m'_1, \ldots, m'_s)$, where

$$m'_j = \begin{cases} 
  m_j + 1 & \text{if } m_j < q, \\
  q & \text{if } m_j = q.
\end{cases}$$

(4.5.4) Lemma

Let $W$ be an irreducible closed subvariety of $X \times \Delta^p$ such that $W$ is in $Z^{q}_{C,m-1}(X,p)$. Then there is an open subset $U_{W,C,m}$ of $U_X$ such that, for each $L \in U_{W,C,m}$, we have

a) $(\pi_{L,X} \times \text{id}_{\Delta^p})_*(W)$ is in $Z^{q}_{C,m-1,\pi_{L,X}}(\mathbb{A}^n,p)$

b) $(\pi_{L,X} \times \text{id}_{\Delta^p})^*((\pi_{L,X} \times \text{id}_{\Delta^p})_*(W)) = W + W'$, with $W'$ effective.

c) $W'$ is in $Z^{q}_{C,m}(X,p)$.

Proof. Let $f: X \to Y$ be a finite morphism of smooth $k$-varieties. Then the maps $(f \times \text{id}_{\Delta^p})_*$, $(f \times \text{id}_{\Delta^p})^*$ give maps of complexes

$$f_*: Z^q(X,*) \to Z^q(Y,*)$$
$$f^*: Z^q(Y,*) \to Z^q(X,*)$$

and $f^*(f_*(Z))$ is equal to $Z + Z'$, with $Z'$ effective if $Z$ is effective. This implies (b), and shows that (c) implies (a).
To prove (a), let \( F \) be a face of \( \Delta^p \). Write \( W \cap (X \times F) \) as a union of irreducible components

\[
W \cap (X \times F) = W_F^1 \cup \ldots W_F^t
\]

and let \( W_F^{i,j} \) be the locally closed subset of \( X \) defined by

\[
x \in W_F^{i,j} \iff \dim_k((x \times F) \cap W_F^i) = j
\]

\[j = 0, 1, \ldots, \dim_k(F); \quad i = 1, \ldots, t.
\]

Let \( C_0 = X, m_0 = q \). We note that \( W \) is in \( \mathcal{Z}^j_{C,m-1}(X,p) \) if and only if the inequalities

\[
\text{codim}_{C_l}(W_F^{i,j} \cap C_l) \geq m_i' + j - \dim_k(F).
\]

(1)

hold for all \( i, j \) and \( l \), where

\[
m_i' = \begin{cases} 
\min(m_l - 1, 0) & l = 1, \ldots, s, \\
q & l = 0.
\end{cases}
\]

For a locally closed subset \( A \) of \( X \), and an \( L \in \mathcal{U}_X \), let \( L^+(A) \) be the closure in \( \pi_{L,X}^{-1}(\pi_{L,X}(A)) \) of \( \pi_{L,X}^{-1}(\pi_{L,X}(A)) - A \). Define \( L^+(W) \) similarly, by taking the closure in \( X \times \Delta^p \) of

\[
(\pi_{L,X} \times \text{id}_{\Delta^p})^{-1}(\pi_{L,X} \times \text{id}_{\Delta^p}(W)) - W.
\]

If we write \( L^+(W) \cap (X \times F) \) as a union of irreducible components,

\[
L^+(W) \cap (X \times F) = \cup_{i'} L^+(W_F^{i'})^\prime.
\]

Then each \( L^+(W)_F^{i'} \) is an irreducible component of \( L^+(W_F^{i'}) \) for some \( i' \); we write this as

\[
i = \nu(i').
\]

If we then define the locally closed subsets \( L^+(W)_F^{i',j} \) for \( L^+(W) \) in a similar fashion to the definition of \( W_F^{i,j} \); one then has the inclusion

\[
L^+(W)_F^{i',j} \subset L^+(W_F^{\nu(i'),j}).
\]

(2)

and each irreducible component of \( L^+(W)_F^{i',j} \) is dense in an irreducible component of \( L^+(W_F^{\nu(i'),j}) \). If we write

\[
(\pi_{L,X} \times \text{id}_{\Delta^p})^* (\pi_{L,X} \times \text{id}_{\Delta^p}_*(W)) = W + W',
\]

then clearly \( L^+(W) \) is the support of \( W' \).
Let $A$ be a locally closed subset of $X$. By the main result of [R], there is a non-empty open subset $U$ of $U_X$ such that, for each $L \in U$, we have

$$L^+(A) \cap (C - (A \cap R_L \cap C))$$

is a proper intersection on $X$, and such that $R_L$ contains no irreducible component of $A \cap C$. In addition, if $Z$ is a component of proper intersection of $L^+(A) \cap C$ then $Z$ is not contained in $R_L$. Thus, if we let $e(A, C)$ denote the maximum among the irreducible components $Z$ of $A \cap C$ of the expression

$$\operatorname{codim}_C(Z) - \operatorname{codim}_X(A)$$

we have

$$e(L^+(A), C) \leq \max(e(A, C) - 1, 0).$$

for all $L \in U$.

If we apply this with $A = W^{i,j}_i$, we find that there is for each $F$, $i$ and $j$ a non-empty open subset $U^{i,j}_F$ of $U_X$ such that, for $L \in U^{i,j}_F$, we have

$$\operatorname{codim}_{C_1}(L^+(W)^{i',j}_F \cap C_l) \geq \min(m'_l + j - \dim_k(F) + 1, \operatorname{codim}_X(W^{i,j}_F))$$

for all $i'$ with $\nu(i') = i$. On the other hand, by (1) for $l = 0$, we have

$$\operatorname{codim}_X(W^{i,j}_F) \geq \dim_k(F) + q - j,$$

so (3) is equivalent to

$$\operatorname{codim}_{C_1}(L^+(W)^{i',j}_F \cap C_l) \geq \min(m'_l + j - \dim_k(F) + 1, \dim_k(F) + q - j)$$

(4)

Noting that

$$m_l = \min(m'_l + 1, q)$$

we see that (4) is equivalent to

$$\operatorname{codim}_{C_1}(L^+(W)^{i',j}_F \cap C_l) \geq m_l + j - \dim_k(F).$$

(5)

Now take $L$ in the intersection of all the $U^{i,j}_F$. As (5) implies

$$\operatorname{codim}_{C_1}(L^+(W)^{i',j}_F \cap C_l) \geq m_l + j - \dim_k(F),$$

for all $i'$, $j$ and $F$, and as $L^+(W)$ is the support of $W'$, we see that $W'$ is in $Z^d_{C,m}(X, p)$, as desired. \qed
**Bloch’s higher chow groups**

*(4.5.5) Finite pull-back*

Let $Y = \mathbb{A}^n$, and suppose we have a finite map

$$f: X \to Y,$$

a collection of locally closed subsets $C = \{C_1, \ldots, C_s\}$ of $X$, and a sequence of integers $m_1, \ldots, m_s$ with $0 \leq m_j \leq q$. We let $\mathcal{Z}_{C,m,f}^q(Y,p)$ be the subgroup of $\mathcal{Z}^q(Y,p)$ generated by the irreducible codimension $q$ subvarieties $W$ of $Y \times \Delta^p$ such that

$$\text{codim}_{C_j \times F}((C_j \times F) \cap (f \times \text{id})^{-1}(W)) \geq m_j.$$

The $\mathcal{Z}_{C,m,f}^q(Y,p)$ form a subcomplex of $\mathcal{Z}^q(Y,*)$, and the map

$$f^*: \mathcal{Z}^q(Y,*) \to \mathcal{Z}^q(X,*)$$

restricts to the map

$$f^*: \mathcal{Z}_{C,m,f}^q(Y,*) \to \mathcal{Z}_{C,m}^q(X,*).$$

We let $m_{\text{max}}$ denote the constant sequence

$$m_{\text{max}} = (q_1, \ldots, q).$$

*(4.5.6) A triangulation*

We have the vertices $v_0^p, \ldots, v_p^p$ of $\Delta^p$, where the vertex $v_j^p$ is given by $t_j = 1$, $t_i = 0$, $i \neq j$. For $i = 0, 1$, $j = 0, \ldots, p$, we let $v_{i,j}^p$ be the point of $\Delta^1 \times \Delta^p$

$$v_{i,j}^p = v_i^1 \times v_j^p; \quad i = 0, 1; \ j = 0, \ldots, p.$$

We let $[n]$ denote the set $\{0, \ldots, n\}$. For each $i = 0, \ldots, p$, we let

$$f_i^p: [p + 1] \to [1] \times [p]$$

be given by

$$f_i^p(j) = \begin{cases} (0, j) & \text{if } 0 \leq j \leq i, \\ (1, j - 1) & \text{if } i + 1 \leq j \leq p + 1. \end{cases}$$

If we let $h^p = \sum_{i=0}^{p} (-1)^i f_i^p$, the $h^p$ form a triangulation of $\Delta^1 \times \Delta^p$:

$$h^p \circ \left( \sum_{i=0}^{p+1} (-1)^i \delta^p_i \right) + \left( \sum_{i=0}^{p} (-1)^i (\text{id} \times \delta^p_i) \right) \circ h_{p-1} = i_1^p - i_0^p,$$

where

$$i^p_1: [p] \to [1] \times [p]$$

$$i^p_1(k) = (j, k).$$
Write $f_i^p = (f_{i,1}^p, f_{i,2}^p)$. We let

$$F_i^p: \Delta^{p+1} = \mathbb{A}^{p+1} \to \Delta^1 \times \Delta^p = \mathbb{A}^{p+1}; \quad i = 0, \ldots, p,$$

be the affine-linear map with

$$F_i^p(v_j^{p+1}) = v_{f_{i,1}(j)}^1 \times v_{f_{i,2}}^p.$$

We call a linear subset $F$ of $\Delta^1 \times \Delta^p$ a face if

$$F = F_i^p(F')$$

for some $i$ and some face $F'$ of $\Delta^{p+1}$.

\textbf{(4.5.7) Definition}

Let $\mathcal{Z}^q_{c,m,f,h}(Y \times \Delta^1, p)$ be the subgroup of $\mathcal{Z}^q(Y \times \Delta^1, p)$ generated by the codimension $q$ subvarieties $W$ of $Y \times \Delta^1 \times \Delta^p$ such that $(\text{id}_Y \times F_i^{p'})^*(W)$ is in $\mathcal{Z}^q_{c,m,f}(Y, p + 1)$ for all $i = 0, \ldots, p$.

Let $F$ be a face of $\Delta^1 \times \Delta^p$, and let $C$ be the disjoint union

$$C = \coprod_{i=1}^s C_i.$$

Let $W$ be an irreducible codimension $q$ subvariety of $Y \times \Delta^p$ such that $W$ is in $\mathcal{Z}^q(Y, p)$, and let $W_F$ be the intersection

$$W_F := p_{13}^{-1}(W) \cap Y \times F \subset Y \times \Delta^1 \times \Delta^p.$$

Let $(G, 1)$ be the pointed affine space $(\mathbb{A}_k^n, 0)$, considered as an algebraic group under addition, and acting on $Y := \mathbb{A}_k^n$ via translation. We use coordinates $x_1, \ldots, x_n$ for $G$ and $y_1, \ldots, y_n$ for $Y$. Let

$$\pi: G - \{0\} \to \mathbb{P}^{n_1}$$

be the canonical map

$$\pi(x_1, \ldots, x_n) = (x_1 : \ldots : x_n).$$

For $x = (x_1, \ldots, x_n) \in G - \{0\}$, the closure in $G$ of fiber $\pi^{-1}(x)$ is canonically isomorphic to $\Delta^1$ via the unique linear map which sends 0 to $v_0^1$, and sends $x$ to $v_1^1$. We write this isomorphism as

$$\phi_x: \Delta^1 \to G.$$

Let

$$i_F: W_F \to Y \times F$$
be the inclusion, let
\[ f_C: C \to Y \]
be the composition of \( f \) with the natural map \( C \to X \), and let
\[ T: G \times C \to Y \]
be the map
\[ T(g, c) = g + f_C(c). \]
Let
\[ p_{F,1}: F \to \Delta^1; \quad p_{F,2}: F \to \Delta^p \]
be the projections.

Consider the diagram
\[
p_{13}^{-1}(W) \quad \downarrow \quad G \times C \times \Delta^1 \times \Delta^p \quad \overset{q}{\longrightarrow} \quad Y \times \Delta^1 \times \Delta^p
\]
where \( q \) is the map
\[ q(g, c, t, \lambda) = (g + i_C(c), t, \lambda). \]
Let
\[ \phi: \Delta^1 \to G \]
be an affine-linear map sending \( v_0^1 \) to 0, and let \( \Phi \) be the map
\[ \Phi: C \times \Delta^1 \times \Delta^p \to G \times C \times \Delta^1 \times \Delta^p \]
\[ \Phi(c, t, \lambda) = (\phi(t), c, t, \lambda). \]
This gives the diagram
\[
\phi_C^*W \quad \longrightarrow \quad p_{13}^{-1}(W) \\
\downarrow \quad \downarrow \\
C \times \Delta^1 \times \Delta^p \quad \overset{q\circ\Phi}{\longrightarrow} \quad Y \times \Delta^1 \times \Delta^p
\]
where \( \phi_C^*W \) and the maps \( \phi_C^*W \to p_{13}^{-1}(W) \) and \( \phi_C^*W \to C \times \Delta^1 \times \Delta^p \) are defined to make the diagram Cartesian. For a face \( F \), we let \( \phi_C^*W_F \) be the intersection
\[ \phi_C^*W_F = \phi_C^*W \cap (C \times F). \]
We let \( F^0 \) be the open subset of \( F \),
\[ F^0 = F - v_0^1 \times \Delta^p, \]
and $\phi_C^* W_F^0$ the open subset of $\phi_C^* W_F$

$$\phi_C^* W_F^0 = \phi_C^* W_F \cap C \times F^0.$$  

(4.5.8) Lemma

There is a non-empty Zariski open subset $U_{W,C}$ of $U$ such that, for each $x \in U_{W,C}$, and each face $F$ of $\Delta^1 \times \Delta^p$, $\phi_x^* W_F^0$ has codimension $q$ on $C \times F^0$, or is empty.

Proof. It suffices to show the existence, for each face $F$, of a non-empty open subset $U_{W,C} F$ of $U_{W,C}$ such that, for each $x \in U_{W,C} F$, $\phi_x^* W_F^0$ has codimension $q$ on $C \times F^0$. We may assume that $F$ is not contained in $v_0^1 \times \Delta^p$. Let $F' = p_{F,2}(F)$. We consider three cases:

a) $p_{F,2}: F \to F'$ is an isomorphism, and $p_{F,1}: F \to \Delta^1$ is surjective.

b) $F = v_1^1 \times F'$.  

c) $F = \Delta^1 \times F'$.

It suffices to handle the case of a single locally closed subset $C$ of $X$; we consider the case (a) first.

By (a), we may identify $F$ with the transpose of the graph of a surjective affine linear map

$$L: F' \to \mathbb{A}^1,$$

where we identify $\mathbb{A}^1$ with $\Delta^1$ via the affine-linear map sending $0$ to $v_0^1$ and $1$ to $v_1^1$. Let

$$F^{00} = L^{-1}(\mathbb{A}^1 - 0),$$

and let

$$\Psi: G \times C \times F^{00} \to Y \times F^0$$

be the map

$$\Psi(x, c, \lambda) = (L(\lambda) \cdot x + f(c), L(\lambda), \lambda).$$

We claim that $\Psi$ is surjective, with fibers of dimension $\dim_k(C)$. Indeed, for $(y, \lambda) \in Y \times F^0$, we have $L(\lambda) \neq 0$. Thus the translates of $f(C)$ by elements of the form $L(\lambda) \cdot x$ cover all of $Y$, and the projection

$$p_2: \Psi^{-1}((y, \lambda)) \to C$$

is a bijection, proving the claim.

Since $W$ is in $Z^q(Y,p)$, $W \cap Y \times F'$ has codimension $q$ on $Y \times F'$. Thus $W_F := p_{13}^{-1}(W) \cap Y \times F$ has codimension $q$ on $Y \times F$, and hence $\Psi^{-1}(W_F)$ has codimension $q$ on $G \times C \times F^{00}$.

Let

$$\Pi: G \setminus \{0\} \times C \times F^{00} \to \mathbb{P}^{n-1}$$

be the map induced by the canonical projection

$$\pi: G \setminus \{0\} \to \mathbb{P}^{n-1}.$$
Since $\Psi^{-1}(W_F)$ has codimension $q$ on $\mathbb{A}^n \times C \times F^0$, it follows that $\Psi^{-1}(W_F) \cap \Pi^{-1}(z)$ has codimension $q$ on $\pi^{-1}(z) \times C \times F^0$ for all $z$ in an open subset $V$ of $\mathbb{P}^{n-1}$. For $x \in G - \{0\}$, the map $\phi_x$ gives an isomorphism of $\Delta^1 - v_0^1$ with $\pi^{-1}(\pi(x))$, and identifies $\Psi^{-1}(W_F) \cap \Pi^{-1}(z)$ with $\phi_x^* W^0_F$, completing the proof in case (a).

The case (b) is similar; we write $F = v_1^1 \times F$, which identifies $F$ with the transpose of the graph of the constant map $L: F' \to \mathbb{A}^1$ with value 1. The same proof as in (a) gives the desired conclusion.

For (c), we have
$$F^0 = (\Delta^1 - v_0^1) \times F';$$
let $\Psi$ be the map
$$\Psi: G \times C \times F^0 \to Y \times F^0$$
$$\Psi(x, c, t, \lambda) = (t \cdot x + f(c), t, \lambda).$$

The argument of (a) shows that $\Psi$ is surjective with fibers of dimension $\dim_k(C)$; continuing the argument by considering the projection
$$\Pi: G - \{0\} \times C \times F^0 \to \mathbb{P}^{n-1}$$
leads to the desired conclusion.

For a pointed map
$$\phi: (\mathbb{A}^1, 0) \to (G, 1)$$
we have the automorphism
$$T^\phi: Y \times \Delta^1 \to Y \times \Delta^1$$
$$T^\phi(x, t) = (\phi(t) + x, t),$$
where we identify $(\Delta^1, v_0^1, v_1^1)$ with $(\mathbb{A}^1, 0, 1)$ as above. For a cycle $W$ on $Y \times \Delta^p$, we let $T^\phi(W)$ be the cycle $(T^\phi \times \text{id})^* (p_{13}^* W)$ on $Y \times \Delta^1 \times \Delta^p$.

Let $i_1$ be the inclusion
$$i_1: Y \to Y \times \Delta^1$$
$$i_1(y) = (y, v_1^1).$$

\textbf{(4.5.9) Lemma}

Let $W$ be an subvariety of $Y \times \Delta^p$ which is in $Z^q_{c, m-1, f}(Y, p)$. Then

i) for each $x \in U_{W, C} T^\phi_x(W)$ is in $Z^q_{c, m, f, h}(Y \times \Delta^1, p)$.

ii) The cycle $i_1^*(W)$ is in $Z^q_{c, m, \text{max}, f}(Y \times \Delta^1, p)$

Proof. Let $F$ be a face of $\Delta^1 \times \Delta^p$. If $F$ is contained in $v_0^1 \times \Delta^p$, then, for each $C \in \mathcal{C}$, or for $C = X$, we have
$$(f \times \text{id})^* (T^\phi_x(W)) \cap (C \times F) = f^* (W) \cap C \times (p_2(F)).$$
As $W$ is in $\mathcal{Z}_{C,m,f}^q(Y,p)$, this intersection has the required codimension on $C \times F$. For all other faces $F$, it follows from (4.5.8) that

$$(f \times \text{id})^*(T_{\phi_s}^*(W)) \cap (C \times F^0)$$

has codimension $q$ on $C \times F^0$. Taking $F$ to be of the form $v_1^1 \times F'$ proves (ii). For the other $F$, $C \times F - C \times F^0$ is equal to $C \times F'$ for some face $F'$ contained in $v_0^1 \times \Delta^p$. As we have already shown that the intersection with $C \times F'$ has the required codimension on $C \times F'$, we have an even better bound for the codimension of

$$(f \times \text{id})^*(T_{\phi_s}^*(W)) \cap (C \times F).$$

This completes the proof.

(4.5.10) The homotopy

We now take $K$ to be transcendental extension of $k$ given by

$$K = k(t_{11}, \ldots, t_{Nn}),$$

and $F$ the transcendental extension of $K$ given by

$$F = K(s_1, \ldots, s_n).$$

Let

$$\pi_t: \mathbb{A}_K^n \to \mathbb{A}_K^n$$

the linear map with matrix

$$
\begin{pmatrix}
  t_{11}, \ldots, t_{1,n} \\
  \vdots \\
  t_{N1}, \ldots, t_{Nn}
\end{pmatrix}
$$

and let

$$\pi_{t,X} X_K \to \mathbb{A}_K^n = Y_K$$

be the restriction to $X_K$. Let

$$\phi_s: (\mathbb{A}_K^1, 0) \to (G_F, 1)$$

be the map

$$\phi_s(z) = z \cdot s.$$ 

Let

$$H_p: \mathcal{Z}_{C,m,f,h}^q(Y, \Delta^1, p) \to \mathcal{Z}_{C,m,f}^q(Y, p + 1)$$

be the map

$$H_p = \sum_{i=0}^p (-1)^i F_i^{p*}.$$
Let \( H^X_{p,C,m} \) denote the composition

\[
Z^q_{C,m-1}(X,p)^{\pi_t,X \circ p^K} Z^q_{C,m-1,\pi_t,X}(Y_K,p) \\
\xrightarrow{T_\phi \circ p_F^*} Z^q_{C,m-1,\pi_t,X,h}(Y_F \times \Delta^1,p) \\
\xrightarrow{H_F} Z^q_{C,m-1,f}(Y_F,p+1)
\]

where \( p_K \) and \( p_F \) are induced by the projections

\[
p_K: X_K \to X, \\
p_F: Y_F \to Y_K.
\]

It follows from (4.5.4) and (4.5.9) that \( H^X_{p,C,m} \) is well-defined, and that the maps \( H^X_{p,C,m} \) give a homotopy between the maps

\[
p_F^* \circ \pi_{t,X}^* \circ \pi_{t,X, *} \circ p_K^*: Z^q_{C,m-1}(X,*) \to Z^q_{C,m-1}(X_F,*),
\]

\[
i_1^* \circ T_{\phi,s}^* \circ p_F^*: Z^q_{C,m-1}(X,*) \to Z^q_{C,m-1}(X_F,*).
\]

In addition, by (4.5.9), the map \( i_1^* \circ T_{\phi,s}^* \circ p_F^* \) factors through the inclusion

\[
Z^q_{C,m_{\text{max}}}(X_F,*) \to Z^q_{C,m-1}(X_F,*).
\]

Finally, it follows from (4.5.4) that the map

\[
p_F^* \circ \pi_{t,X}^* \circ \pi_{t,X,*} \circ p_K^* - \text{id}
\]

factors through the inclusion

\[
Z^q_{C,m}(X_F,*) \to Z^q_{C,m-1}(X_F,*).
\]

Thus, we have shown

(4.5.11) Lemma

The base extension from \( k \) to \( F \) gives a homotopically trivial map

\[
\frac{Z^q_{C,m-1}(X,*)}{Z^q_{C,m}(X,*)} \to \frac{Z^q_{C,m-1}(X_F,*)}{Z^q_{C,m}(X_F,*)}
\]
**Theorem (4.5.12)**

Let \((X, f)\) be in \(\text{Sm}_k\), with \(X\) affine. Then the inclusion

\[
\mathcal{Z}^q(X, \ast)_f \to \mathcal{Z}^q(X, \ast)
\]

is a quasi-isomorphism.

**Proof.** Let \(K\) be a finite extension of \(k\); we then have the base-extension and norm maps

\[
p^*_K: \mathcal{Z}^q(X, \ast)_f \to \mathcal{Z}^q(X, \ast)_f^K,
p_K*: \mathcal{Z}^q(X, \ast)_f^K \to \mathcal{Z}^q(X, \ast)_f,
\]

with

\[
p_K* \circ p^*_K = [K:k] \cdot \text{id}.
\]

If \(k\) is a finite field, there exist infinite pro-\(l\) extensions of \(k\) for each prime \(l\) different from \(\text{char}(k)\); using (1), we may assume that \(k\) is infinite.

From (4.5.2) and an elementary induction, it suffices to show that

\[
\frac{\mathcal{Z}^q_{C,m-1}(X, \ast)}{\mathcal{Z}^q_C(X, \ast)}
\]

is acyclic for all choices of \(C\) and \(m\). By (4.5.11), the map

\[
\frac{\mathcal{Z}^q_{C,m-1}(X, \ast)}{\mathcal{Z}^q_{C,m}(X, \ast)} \to \frac{\mathcal{Z}^q_{C,m-1}(X_F, \ast)}{\mathcal{Z}^q_{C,m}(X_F, \ast)}
\]

is zero on homology. On the other hand, since \(F\) is a pure transcendental extension of the infinite field \(k\), an elementary specialization argument shows that the above map is injective on homology, hence the complex (2) is acyclic, as desired. \(\square\)

**Corollary (4.5.13)**

Let \((X, f)\) and \((Y, g)\) be in \(\mathcal{L}(\text{Sm}_k)\), with \(Y\) affine. Let \(p: X \times_k Y \to X\) be the projection. Then the natural map

\[
\text{id}^*: p_*(\mathcal{Z}^q_{X \times_k Y/k}(\ast)_f \times g) \to p_*(\mathcal{Z}^q_{X \times_k Y/k}(\ast))
\]

is a quasi-isomorphism of complexes of sheaves on \(X\).

**Proof.** Let \(x\) be a point of \(X\). The stalk \([p_*(\mathcal{Z}^q_{X \times_k Y/k}(\ast)_f \times g)]_x\) is the direct limit of the complexes

\[
\mathcal{Z}^q(U \times_k Y, \ast)_{j^*f \times g}
\]
Bloch’s higher chow groups

over affine open neighborhoods \( j: U \to X \); we have the similar description of the stalk
\([p_*(3^q_{X \times_k Y/k(*)})_x]\). By (4.5.12), the maps

\[ Z^q(U \times_k Y, *) \xrightarrow{j^* f \times g} Z^q(U \times_k Y, *) \]

is an isomorphism for all affine \( U \), whence the result.

4.6. Motivic cohomology and the higher Chow groups

We now verify the criteria (4.3.1.2) and (4.4.1) in case \( S \) is smooth, essentially of finite type, and of dimension \( \leq 1 \) over a field \( k \); if \( \text{char}(k) = 0 \), we take \( R = \mathbb{Z} \), while if \( \text{char}(k) = p > 0 \), we take \( R = \mathbb{Q} \).

(4.6.1)

We first consider the case \( S = \text{Spec}(k) \) with \( \text{char}(k) = 0 \). Let \( X \) be a smooth, quasi-projective \( k \)-variety. From the localization theorem [B2], the natural map

\[ Z^q(X, *) \to M^q_{X/k}(*) \]

is a quasi-isomorphism. From the homotopy theorem [B], the map

\[ p^*_i: Z^q(X, *) \to Z^q(X \times_k \mathbb{A}^1, *) \]

is a quasi-isomorphism.

For \( \tilde{X} \) a closed subset of \( X \), with complement \( j: U \to X \), we let \( Z^q_{\tilde{X}}(X, *) \) denote the cone

\[ Z^q_{\tilde{X}}(X, *) := \text{Cone}(j^*: Z^q(X, *) \to Z^q(X - Z, *))[-1]. \]

If \( Z \) is a smooth subvariety of \( X \), of codimension \( d \), and \( \tilde{Z} \) is a closed subset of \( Z \), the localization theorem [B2] implies that the inclusion \( i_Z: Z \to X \) induces a quasi-isomorphism

\[ i_Z^*: Z^q_{\tilde{Z}}(Z, *) \to Z^q_Z(X, *) \]

(4.6.2) Proposition

Let \( k \) be a field of characteristic zero. Then the conditions (4.3.1.2) and (4.4.1), with \( R = \mathbb{Z} \), are satisfied for \( S = \text{Spec}(k) \), \( V = \text{Sm}_k \). If \( k \) is a field of characteristic \( p > 0 \), then the conditions (4.3.1.2) and (4.4.1), with \( R = \mathbb{Q} \), are satisfied for \( S = \text{Spec}(k) \), \( V = \text{Sm}_k \).

Proof. First assume that \( \text{char}(k) = 0 \). The Gysin morphism condition (4.3.1.2)(iii) follows from (4.6.1.1) and (4.6.1.3). Using (4.6.1.3) and (4.6.1.1), one reduces the homotopy condition (4.3.1.2)(i) to the usual homotopy property (4.6.1.2), together with the localization property (4.6.1.1). The conditions (4.3.1.2)(ii) and (4.4.1) follow in a similar fashion from (4.6.1.1)-(4.6.1.3), together with (4.5.13).
If \( \text{char}(k) = p > 0 \), the map (4.6.1.2) is still a quasi-isomorphism. We have shown in [L] that the maps (4.6.1.1) and (4.6.1.3) are quasi-isomorphisms, after tensoring with \( \mathbb{Q} \). The same argument as above verifies (4.3.1.2) and (4.4.1) for \( R = \mathbb{Q} \).

**4.6.3 The case of curves**

Let \( p_{S}: S \to \text{Spec}(k) \) be smooth, essentially of finite type and of dimension one over a field \( k \), and let \((X, f)\) be in \( \mathcal{L}(\text{Sm}_{S}) \). The functor “compose with \( p_{S} \)” gives the functor

\[
p_{S*}: \text{Sm}_{S} \to \text{Sm}_{k}
\]

inducing the functor

\[
p_{S*}: \mathcal{L}(\text{Sm}_{S}) \to \mathcal{L}(\text{Sm}_{k})
\]

We usually ignore the \( p_{S*} \) in the notion, and simply consider an object \((X, f)\) of \( \mathcal{L}(\text{Sm}_{S}) \) as an object of \( \mathcal{L}(\text{Sm}_{k}) \). In particular, we have the natural inclusion of complexes of sheaves on \( X \)

\[
\iota_{S/k}: \mathfrak{Z}_{X/k/S}(\ast)_{f} \to \mathfrak{Z}_{X/k}(\ast)_{f}
\]

More generally, for each \((Y, g)\) in \( \mathcal{L}(\text{Sm}_{k}) \), we have the inclusion of complexes of sheaves on \( X \):

\[
\iota_{S/k}: p_{*}(\mathfrak{Z}_{X \times k Y/S}(\ast)_{f \times k g}) \to p_{*}(\mathfrak{Z}_{X \times k Y/k}(\ast)_{f \times k g})
\]

(4.6.4) **Lemma**

The map (4.6.3.1) is a quasi-isomorphism.

**Proof.** Let \( x \) be a point of \( X \), and take

\[
W \in p_{*}(\mathfrak{Z}_{X \times k Y/k}(\ast)_{f \times k g})(U)
\]

for some affine open neighborhood \( j: U \to X \) of \( x \). Let \( p_{X}: X \to S \) be the structure map, and let \( s = p_{X}(x) \). Let

\[
i_{U,s}: U_{s} \to U
\]

be the inclusion, where \( U_{s} \) is the fiber of \( U \) over \( s \). If \( W \) is in the subcomplex

\[
p_{*}(\mathfrak{Z}_{X \times k Y/S}(\ast)_{f \times k g})(U),
\]

then \( i_{U,s}^{*}(W) \) is defined and in

\[
p_{*}(\mathfrak{Z}_{U_{s} \times k Y/k(s)}(\ast)_{f_{s} \times k g})(U_{s}) = \mathfrak{Z}_{U_{s} \times k Y/k(s), \ast}(U_{s} \times k Y/k(s), f_{s} \times k g),
\]

where \( f_{s} \) is the map \( f \times S \text{id}_{s} \). Conversely, suppose that the cycle \( i_{U,s}^{*}(W) \) is defined and is in \( \mathfrak{Z}_{U_{s} \times k Y/k(s), \ast}(U_{s} \times k Y/k(s), f_{s} \times k g) \). Since \( S \) has dimension one, this implies that \( W \) is equi-dimensional over an open neighborhood of \( s \) in \( S \); similarly, all the necessary intersections of \( W \) with faces, and all the various pull-backs of \( W \) are equi-dimensional over a neighborhood.
of $s$ in $S$. Thus $W$ is in $p_*(\mathcal{Z}_X^q Y/S(\{f \times k\}))(V)$ for some neighborhood $V \subset U$ of $x$. Letting

$$i_s : X_s \rightarrow X$$

be the inclusion, we have just shown the identity on the stalks

$$[p_*(\mathcal{Z}_X^q Y/k(\{f \cup i_s\} \times k))] x = [p_*(\mathcal{Z}_X^q Y/S(\{f \times k\}))) x.$$

As the inclusion

$$[p_*(\mathcal{Z}_X^q Y/k(\{f \cup i_s\} \times k))] x \rightarrow [p_*(\mathcal{Z}_X^q Y/k(\{f \times k\}))) x$$

is a quasi-isomorphism by (4.5.13), the proof is complete. \qed

(4.6.5) Proposition

Let $k$ be a field, and let $S$ be a smooth $k$-scheme, essentially of finite type and of dimension one over $k$. If $\text{char}(k) = 0$, then the conditions (4.3.1.2) and (4.4.1), with $R = \mathbb{Z}$, are satisfied for $S$, with $V = \text{Sm}_S$. If $k$ is a field of characteristic $p > 0$, then the conditions (4.3.1.2) and (4.4.1), with $R = \mathbb{Q}$, are satisfied for $S$, with $V = \text{Sm}_S$.

Proof. This follows from (4.6.4) and (4.6.2). \qed

(4.6.6) Theorem

Let $k$ be a field, and let $S$ be a smooth $k$-scheme, essentially of finite type and of dimension zero or one over $k$. Let $V$ be a full subcategory of $\text{Sm}_S$ such that the conditions (i)-(iii) of (2.1.3) are satisfied. If $\text{char}(k) = 0$, take $R = \mathbb{Z}$; if $\text{char}(k) = p > 0$, take $R = \mathbb{Q}$. Then

i) the cycle class map

$$\text{cl}(\Gamma) : \mathcal{C}(\Gamma) \otimes R \rightarrow \text{Hom}_{\mathcal{D}M(V)}(1, \Gamma) \otimes R$$

is an isomorphism for all $\Gamma$ in $\mathcal{D}M(V)$.

ii) the cycle class map

$$\text{cl}_{X}^{q,p} : \mathcal{C}^q(X, 2q - p) \otimes R \rightarrow H^p(X, R(q))$$

is an isomorphism for all $X$ in $V$.

iii) The natural map

$$\mathcal{Z}_q(X/k, *) \rightarrow \mathfrak{M}_q(X/S, *)$$

induces an isomorphism

$$\text{CH}^q_{\text{naive}}(X/k, p) \otimes R \rightarrow \mathcal{C}^q(X/S, p) \otimes R$$
for all $X$ in $\mathcal{V}$. In addition, the group $\text{CH}^q_{\text{naive}}(X/k, p)$ is Bloch’s higher Chow group $\text{CH}^q(X, p)$.

iv) For $X$ in $\mathcal{V}$ there is a natural isomorphism

$$K_{2q-p}(X)^{(q)} \to H^p(X, \mathbb{Q}(q)).$$

Here $K_n(X)^{(q)}$ denotes the weight-$q$ eigenspace of the Adams operations on $K_n(X) \otimes \mathbb{Q}$.

Proof. The first statement follows from (4.4.8), (4.6.2) and (4.6.5); the second follows from the first and the definition of the map $\text{cl}^q_{X,p}$, the higher Chow group $\text{CH}^q(X, p)$, and the motivic cohomology group $H^p(X, R(q))$ as

$$H^p(X, R(q)) = \text{Hom}_{DM(\mathcal{V})}(1, \mathbb{Z}_X(q)[p]) \otimes R$$

$$\text{CH}^q(X, p) = \text{CH}(\mathbb{Z}_X(q)[2q-p])$$

$$\text{cl}^q_{X,p} = \text{cl}(\mathbb{Z}_X(q)[2q-p]).$$

The first part of (iii) follows from the localization theorem of [B2], in case $\text{char}(k) = 0$, or the $\mathbb{Q}$-analog in characteristic $p > 0$ given in [L]. The second part follows from (4.1.11).

For (iv), we have shown in [L] that there is a natural isomorphism of $K_n(X)^{(q)}$ with $\text{CH}^q(X, n) \otimes \mathbb{Q}$, for $X$ a smooth, quasi-projective variety over a field. Combining this with (iii) completes the proof.

(4.6.7) Remark

The requirement that the coefficient ring $R = \mathbb{Q}$ in case $\text{char}(k) > 0$ arises from the use of resolution of singularities in [B2]; if one has resolution of singularities for varieties over $k$, one can replace the coefficient ring $R$ with $\mathbb{Z}$ in all the results in this section.