MATH 500 — AUGUST 2017

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. (a) Let \( u_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \) denote the upper triangular nilpotent matrix with 1s just above the diagonal and 0s elsewhere. Show that if \( c \) is any nonzero complex number and \( I_n \) is the \( n \times n \) identity matrix, then \((cI_n + u_n)^k \neq I_n\) for all \( k > 0 \).

(b) Let \( GL_n(\mathbb{C}) \) denote the group of invertible \( n \times n \) matrices with complex coefficients (with matrix multiplication as the group operation). Prove that for every \( k > 0 \), if \( \Phi : \mathbb{Z}/k\mathbb{Z} \to GL_n(\mathbb{C}) \) is any group homomorphism, there exists some \( g \in GL_n(\mathbb{C}) \) such that \( g\Phi(m)g^{-1} \) is a diagonal matrix for all \( m \in \mathbb{Z}/k\mathbb{Z} \).

(c) Prove by example that the conclusion of part (b) can fail if \( GL_n(\mathbb{C}) \) is replaced by \( GL_n(F) \) for appropriate choices of integers \( k \) and \( n \) and finite field \( F \).

2. Must a group of order \( 3 \cdot 5 \cdot 7 \) be solvable? Justify your answer.

3. Let \( A = \begin{pmatrix} -2 & -2 & -1 \\ 0 & -4 & -1 \\ 0 & +4 & 0 \end{pmatrix} \). Make \( V = \mathbb{C}^3 \) into a \( \mathbb{C}[x]- \)module by \( f(x)v := f(A) \cdot v \) (matrix multiplication) for \( f(x) \in \mathbb{C}[x] \), \( v \in V \).

Find an elementary divisor decomposition of the module \( V \). Justify your answer.

4. Consider the ring \( R = \mathbb{C}[x, y, z]/(x^2 - xy) \). Show that \( R \) is not a UPD.

5. Let \( K = \mathbb{Q}(\omega) \) where \( \omega = e^{2\pi i/17} \).

(a) Prove that \( K \) contains a unique subfield \( L \) such that \( [L : \mathbb{Q}] = 8 \).

(b) Is \( L \) Galois over \( \mathbb{Q} \)? Justify.